AIRCRAFT ANALYTIC GEOMETRY

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Douglas A-20 Havoc attack bomber.

(*Frontispiece.*)

Aircraft Analytic Geometry

Applied to Engineering, Lofting, and Tooling

BY

J. J. APALATEGUI

Tooling Project Supervisor, Lofting Departments
Douglas Aircraft Company, Inc.

AND

L. J. ADAMS

Head of Department of Mathematics
Santa Monica Junior College

FIRST EDITION
SECOND IMPRESSION

McGRAW-HILL BOOK COMPANY, INC.
NEW YORK AND LONDON
1944

AIRCRAFT ANALYTIC GEOMETRY

PRINTED IN THE UNITED STATES OF AMERICA

THE MAPLE PRESS COMPANY, YORK, PA.

To

G. A. HUGGINS

Plant Manager, Long Beach Plant

Douglas Aircraft Company, Inc.

Who, as the first Director of Tooling for this company, initiated and encouraged the application of analytic geometry to lofting and tooling methods

PREFACE

This book is based on the notes written by J. J. Apalategui for presentation in his class in aircraft analytic geometry, conducted in the Douglas Santa Monica plant at intervals during the past four years for the training of loftsmen, engineers, and tool designers, and recently taught in the ESMWT Tooling Training Program at the University of California at Los Angeles.

The material has been carefully organized and arranged so that designing and tooling engineers in the industry may, through home study of this work, gain a working knowledge of a basic mathematical concept that will greatly facilitate geometric and trigonometric computations in their fields.

The book constitutes a new approach to a certain class of problems which arise in the engineering, lofting, and tooling of airplanes. The approach is mathematical and is based on the principles of plane and solid analytic geometry. Some of the ideas treated are calculation of the true length of a line segment, true angle between two lines, true distance from a point to a line, shortest distance between two lines in space, true angle between a line and a plane, true angle between two planes, true distance from a point to a plane, equations of lines and planes, revolution of a point about a line, mathematical calculation of single-canted and double-canted ribs, rotation of axes in one plane, rotation of axes through both incidence and dihedral, graphical and mathematical analysis of conics, locating points and determining angles and dimensions necessary in engineering, tool designing, layout, and jig building.

The book is intended for use by men in the lofting, tool designing, and jig-building departments and will also be very useful for men in the layout and development groups of the engineering department. It will be of particular interest to students and instructors of college and ESMWT descriptive geometry courses, since it explains methods which simplify the mathematical checking of layouts made by descriptive geometry. Since the application of this method is new to the industry, it should be welcomed

by all technical men for reasons of their interest in progressive developments.

The methods outlined in this book have proved invaluable to the tooling division of the Douglas Aircraft Company, Inc. They have initiated an exactitude in tool design and tool fabrication by mathematical analysis which has aided in the evolution from cut-to-fit and drill-to-match methods to those of mass production which permit the manufacture of accurately formed detail parts that arrive at the final assembly line for installation without further rework.

J. J. Apalategui,
L. J. Adams.

Los Angeles, Calif.,
Santa Monica, Calif.,
January, 1944.

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FOREWORD

This book is based upon ideas and mathematical methods developed in the tooling division of the Douglas Aircraft Company, Inc., by J. J. Apalategui, tooling project supervisor in charge of lines layouts and lofting procedures. It presents an exact approach to a large class of geometrical problems which arise in the lofting, engineering, and tooling of airplanes—an approach which has contributed largely to the ever-increasing demands of the industry for precision manufacturing.

The application of analytic geometry as a precise method in the location of fundamental points, lines, and planes and in the calculation and rotation of angles was initiated by the author in 1937 during the tooling of the prototype C-54 and developed to its highest point on the B-19 superbomber. The complete analysis and definition of loft lines layouts by the equations of conics were introduced in 1940.

Credit for the development and refinement of this work is due to the initiative, ability, and resourcefulness of Mr. Apalategui and his associates in the tooling division.

The selection and organization of the material included in this book are largely the work of L. J. Adams, head of the department of mathematics, Santa Monica Junior College, who also has made valuable contributions to the chapters dealing with the mathematical analysis of curves used in lofting.

The authors, in presenting this difficult subject in a practical, simplified, and self-explanatory form, have unquestionably made a major contribution to the entire industry and to the war effort.

Although this text contains all the essential material, Mr. Apalategui is still conducting and encouraging research along lines which transcend even the limits of this presentation.

A. W. Davies,
Superintendent of Tooling
Santa Monica Plant
Douglas Aircraft Company, Inc

Santa Monica, Calif.,
January, 1944.

GREEK LETTERS

		Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	XI
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ s	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

AIRCRAFT ANALYTIC GEOMETRY

CHAPTER 1

TRIGONOMETRY

This chapter constitutes a brief review of trigonometry. Particular emphasis is placed on those topics of trigonometry which find applications in the succeeding chapters of this book. The definitions of the trigonometric functions, interpolation, inverse interpolation, the solution of triangles, and the fundamental trigonometric identities are essential to the work that follows this chapter.

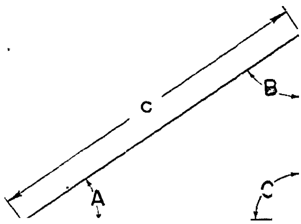


FIG. 1.1.

1.1. Definitions. In the right triangle of Fig. 1.1 the side a is the opposite side, b is the adjacent side, and c is the hypotenuse. The definitions of the six trigonometric functions and their abbreviations are

$$\text{sine } A = \frac{\text{opposite side}}{\text{hypotenuse}}. \qquad \sin A = \frac{a}{c}$$

$$\text{cosine } A \quad \frac{\text{adjacent side}}{\text{hypotenuse}}. \qquad \cos A$$

$$\text{tangent } A = \frac{\text{opposite side}}{\text{adjacent side}}. \qquad \tan A = \frac{a}{b}$$

$$\text{cotangent } A = \frac{\text{adjacent side}}{\text{opposite side}}. \qquad \cot A = \frac{b}{a}$$

$$\text{secant } A = \frac{\text{hypotenuse}}{\text{adjacent side}}. \qquad \sec A = \frac{c}{b}$$

$$\text{cosecant } A = \frac{\text{hypotenuse}}{\text{opposite side}}. \qquad \csc A =$$

1.2. Unit sides. If one side is taken to be unity (one unit long), then the diagrams in Fig. 1.2 result. From Figs. 1.1 and

1.2 several useful relations between the trigonometric functions can be obtained.

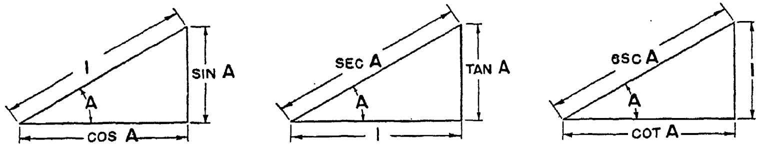


FIG. 1.2.

$$\frac{\sin A}{\cos A} = \tan A. \qquad \frac{\tan A}{\sec A} = \sin A. \qquad \frac{\csc A}{\cot A} = \sec A.$$

$$\frac{\cos A}{\sin A} = \cot A. \qquad \frac{\sec A}{\tan A} = \csc A. \qquad \frac{\cot A}{\csc A} = \cos A.$$

$$\cos A \tan A = \sin A. \qquad \sec A \sin A = \tan A. \qquad \cot A \sec A = \csc A.$$

$$\frac{}{\sin A} = \csc A. \qquad \frac{1}{\cos A} = \sec A. \qquad \frac{1}{\tan A} = \cot A.$$

$$\frac{1}{\cot A} = \tan A. \qquad \frac{1}{\sec A} = \cos A. \qquad \frac{1}{\csc A} = \sin A.$$

$$\sin^2 A + \cos^2 A = 1. \qquad \sec^2 A = 1 + \tan^2 A. \qquad \csc^2 A = 1 + \cot^2 A.$$

$$\cos A + \sin A \tan A = \sec A.$$

1.3. Solving right triangles when a side and an acute angle are given. A right triangle consists of six parts, three sides and three angles. One of the angles is always 90°. The numerical values of the six trigonometric functions are tabulated for each degree and minute. When one of the acute angles and a side are given, the other two sides of the right triangle can be determined by the use of these tables.

Example 1. In Fig. 1.1, let $c = 12$ and $A = 23°$. Find a. From the tables, $\sin 23° = 0.39073$.

$$a = c \sin A.$$
$$a = (12)(0.39073) \quad 4.689.$$

Example 2. In Example 1 find b.

$$\cos 23° = 0.92050.$$
$$b = c \cos A.$$
$$b = (12)(0.92050) = 11.046.$$

Example 3. In Fig. 1.1, let $b = 16$ and $A = 35°$. Find a.

$$\tan 35° = 0.70021.$$
$$a = b \tan A.$$
$$a = (16)(0.70021) = 11.203.$$

Example 4. In Example 3 find c.

$$\cos 35° = 0.81915.$$
$$c = b \sec A = \frac{b}{\cos A}.$$
$$c = \frac{16}{0.81915} = 19.532.$$

Example 5. In Fig. 1.1, let $a = 21$ and $A = 17°$. Find b.

$$\cot 17° = 3.2709.$$
$$b = a \cot A.$$
$$b = (21)(3.2709) = 68.689.$$

Example 6. In Fig. 1.1, let $a = 64$ and $A = 38°$. Find c.

$$\sin 38° = 0.61566.$$
$$a = c \sin A.$$
$$c = \frac{a}{\sin A}.$$
$$c = \frac{64}{0.61566} = 103.953.$$

Example 7. In Fig. 1.1, let $b = 8$ and $A = 20°$. Find c.

$$\cos 20° = 0.93969.$$
$$b = c \cos A.$$
$$c = \frac{b}{\cos A}.$$
$$c = \frac{8}{0.93969} = 8.513.$$

Example 8. In Fig. 1.1, let $a = 36$ and $A = 40°$. Find b.

$$\tan 40° = 0.83910.$$
$$a = b \tan A.$$
$$b = \frac{a}{\tan A}.$$
$$b = \frac{36}{0.83910} = 42.903.$$

Exercises (see Fig. 1.1)

1. $c = 24$, $A = 15°$. Find a.
2. $c = 72$, $A = 15°$. Find b.
3. $b = 72$, $A = 28°$. Find a.
4. $b = 72$, $A = 28°$. Find c.
5. $a = 37$, $A = 19°$. Find b.
6. $a = 54$, $A = 31°$. Find c.
7. $b = 18$, $A = 43°$. Find c.
8. $a = 66$, $A = 22°$. Find b.

1.4. Interpolation. When the angle is given in degrees, minutes, and seconds, the value of the function can be determined by interpolation.

Example 1. Find sin 17°23′12″.

$$\begin{aligned} \sin 17°24' &= 0.29904. \\ \sin 17°23' &= 0.29876. \\ \text{Tabular difference} &= 28. \\ \tfrac{12}{60} \times 28 &= 5.6 = 6. \\ \sin 17°23'12'' &= 0.29876 + 0.00006. \\ &= 0.29882. \end{aligned}$$

Example 2. Find cos 17°23′12″.

$$\begin{aligned} \cos 17°23' &= 0.95433. \\ \cos 17°24' &= 0.95424. \\ \text{Tabular difference} &= 9. \\ \tfrac{12}{60} \times 9 &= 1.8 = 2. \\ \cos 17°23'12'' &= 0.95433 - 0.00002 = 0.95431. \end{aligned}$$

The sine is an increasing function, and the cosine is a decreasing function for the angles between 0° and 90°. The tangent and secant are increasing functions, and the cotangent and cosecant are decreasing functions for the angles between 0° and 90°. The method for interpolating the increasing functions is as shown in Example 1 above, and the method for decreasing functions is as in Example 2.

Exercises

1. Find sin 18°15′45″.
2. Find cos 28°16′20″.
3. Find tan 19°25′15″.
4. Find sin 34°18′16″.
5. Find cos 30°46′37″.
6. Find tan 8°15′51″.

1.5. Functions of complementary angles.

$$\begin{aligned}\sin(90^\circ - A) &= \cos A.\\ \cos(90^\circ - A) &= \sin A.\\ \tan(90^\circ - A) &= \cot A.\\ \cot(90^\circ - A) &= \tan A.\\ \sec(90^\circ - A) &= \csc A.\\ \csc(90^\circ - A) &= \sec A.\end{aligned}$$

Example 1. Find sin 75°16′.

$$\begin{aligned}90^\circ - 75^\circ16' &= 14^\circ44'.\\ \sin 75^\circ16' &= \cos 14^\circ44'.\\ \therefore \sin 75^\circ16' &= 0.96712.\end{aligned}$$

Example 2. Find cos 58°42′.

$$\begin{aligned}90^\circ - 58^\circ42' &= 31^\circ18'.\\ \cos 58^\circ42' &= \sin 31^\circ18'.\\ \cos 58^\circ42' &= 0.51952.\end{aligned}$$

Example 3. Find tan 81°7′.

$$\begin{aligned}90^\circ - 81^\circ7' &= 8^\circ53'.\\ \tan 81^\circ7' &= \cot 8^\circ53'.\\ \tan 81^\circ7' &= 6.3980.\end{aligned}$$

Exercises

1. Find sin 86°15′.
2. Find cos 48°52′.
3. Find tan 81°35′.
4. Find cot 46°17′.
5. Find sin 52°16′31″.
6. Find cos 62°48′52″.
7. Find tan 77°16′45″.

1.6. Inverse Interpolation. When the function is given, the angle can be determined by inverse interpolation if the function is not in the tables.

Example 1. Given $\sin A = 0.18942$. Find A.

$$\begin{aligned}\sin 10^\circ56' &= 0.18967.\\ \sin A &= 0.18942.\\ \sin 10^\circ55' &= 0.18938.\\ \text{Tabular difference} &= 29.\\ \tfrac{4}{29} \times 60'' &= 8''.\\ A &= 10^\circ55'8''.\end{aligned}$$

Example 2. Given $\cos A = 0.98014$. Find A.

$$
\begin{aligned}
\cos 11°26' &= 0.98016.\\
\cos A &= 0.98014.\\
\cos 11°27' &= 0.98010.\\
\text{Tabular difference} &= 6.\\
\tfrac{4}{6} \times 60'' &= 40''.\\
60'' - 40'' &= 20''.\\
A &= 11°26'20''.
\end{aligned}
$$

The increasing functions are calculated as in Example 1, and the decreasing functions as in Example 2.

Exercises

1. Given $\sin A = 0.61650$. Find A
2. Given $\cos A = 0.76971$. Find A
3. Given $\tan A = 0.75548$. Find A
4. Given $\sin A = 0.76631$. Find A
5. Given $\cos A = 0.69395$. Find A
6. Given $\tan A = 1.1286$. Find A.
7. Given $\cot A = 1.1842$. Find A.
8. Given $\csc A = 1.0892$. Find A.
9. Given $\sec A = 1.2168$. Find A.
10. Given $\sec A = 1.5155$. Find A.

1.7. Solving right triangles when two sides are given. When two sides are given, the acute angles can be calculated by using the tables.

Example 1. In Fig. 1.1, let $a = 12$ and $b = 7$. Find A.

$$
\begin{aligned}
\tan A &= \tfrac{12}{7} = 1.7143.\\
\tan 59°45' &= 1.7147.\\
\tan A &= 1.7143.\\
\tan 59°44' &= 1.7136.\\
\text{Tabular difference} &= 11.\\
\tfrac{7}{11} \times 60'' &= 38''.\\
A &= 59°44'38''.
\end{aligned}
$$

Example 2. In Fig. 1.1, let $a = 15$ and $c = 32$. Find A.

$$
\begin{aligned}
\sin A &= \tfrac{15}{32} = 0.46875.\\
\sin 27°58' &= 0.46896.\\
\sin A &= 0.46875.\\
\sin 27°57' &= 0.46870.\\
\text{Tabular difference} &= 26.\\
\tfrac{5}{26} \times 60'' &= 12''.\\
A &= 27°57'12''.
\end{aligned}
$$

Example 3. In Fig. 1.1, let $b = 17$ and $c = 32$. Find A.

$$\cos A = \tfrac{17}{32} = 0.53125.$$
$$\cos 57°54' = 0.53140.$$
$$\cos A = 0.53125.$$
$$\cos 57°55' = 0.53115.$$
$$\text{Tabular difference} = 25.$$
$$\tfrac{10}{25} \times 60'' = 24''.$$
$$60'' - 24'' = 36''.$$
$$A = 57°54'36''.$$

Exercises (see Fig. 1.1)

1. $a = 43$, $b = 21$. Find A.
2. $b = 17$, $c = 24$. Find A.
3. $b = 8$, $c = 29$. Find A.
4. $a = 14$, $c = 42$. Find A.

1.8. Applications. The ideas and principles of the preceding articles can be used to solve many types of problems that arise in the lofting and related departments. Some of these, which bear directly upon the chapters to follow, are illustrated in these examples and exercises.

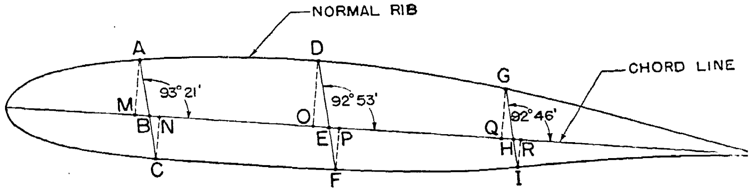


FIG. 1.3.

Example 1. In Fig. 1.3 the line segment AC is the front spar trace on a normal rib plane with B as the intersection with the chord, the line segment DF is the center spar trace on the normal rib plane with E as the intersection with the chord, and the line segment GI is the rear spar trace on the normal rib plane with H as the intersection with the chord. The following dimensions are given:

$$AB = 15.128. \quad BC = 8.140. \quad DE = 16.976.$$
$$EF = 10.013. \quad GH = 8.798. \quad HI = 5.439.$$

(*a*) Find the length of MA.

$$\angle ABM = 180° - 93°21' = 86°39'.$$
$$\frac{MA}{AB} = \sin 86°39'.$$
$$MA = (15.128)(0.99829).$$
$$MA = 15.102.$$

(*b*) Find the length of NC.

$$\begin{aligned} \angle NBC &= 86°39'. \\ \frac{NC}{BC} &= \sin\ 86°39'. \\ NC &= (8.140)(0.99829). \\ NC &= 8.126. \end{aligned}$$

(*c*) Find the length of MB.

$$\begin{aligned} \frac{MB}{AB} &\quad \cos\ 86°39'. \\ MB &= (15.128)(0.05844). \\ MB &= 0.884. \end{aligned}$$

(*d*) Find the length of NB.

$$\begin{aligned} \frac{NB}{BC} &= \cos\ 86°39'. \\ NB &= (8.140)(0.05844). \\ NB &= 0.476. \end{aligned}$$

Exercises

Find the lengths of

1. OD.
2. PF.
3. OE.
4. EP.
5. QG.
6. RI.
7. QH.
8. RH.

Example 2. The cosine of the true angle between two lines is 0.98717. Find the angle.

$$\begin{aligned} \cos 9°11' &= 0.98718. \\ \cos\theta &= 0.98717. \\ \cos 9°12' &= 0.98714. \\ \text{Tabular difference} &= 4. \\ \tfrac{3}{4} \times 60'' &= 45''. \\ 60'' - 45'' &= 15''. \\ \theta &= 9°11'15''. \end{aligned}$$

Example 3. In the plan view of a chord plane wing, the semispan is 600 in. and the angle of sweepback of the leading edge is 4°30′. Find the length of the leading edge in this view. The diagram is similar to Fig. 1.1, where b is the semispan, c is the leading edge, and angle A is the sweepback angle.

$$\begin{aligned} \frac{c}{b} &= \sec A. \\ c &= (600)(1.0031). \\ c &= 601.86. \end{aligned}$$

Example 4. The cosine of the true angle between the center line of the flap hinge and a normal to the plane of a vertical rib is 0.99717. The true angle between the center line and the vertical rib is the complement of the true angle between the center line and the normal to the rib plane. Find the true angle between the center line and the plane of the vertical rib.

$$\begin{aligned}
\cos 4°18' &= 0.99719.\\
\cos A &= 0.99717.\\
\cos 4°19' &= 0.99716.\\
\text{Tabular difference} &= 3.\\
\tfrac{1}{3} \times 60'' &= 20''.\\
60'' - 20'' &= 40''.\\
A &= 4°18'40''.\\
90° - 4°18'40'' &= 85°41'20''.
\end{aligned}$$

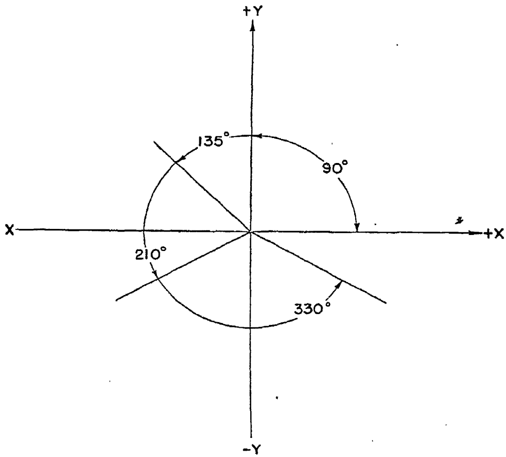


FIG. 1.4.

Another method would be to make use of the fact that $\sin (90° - A) = \cos A$.

$$\begin{aligned}
\sin 85°42' &= 0.99719.\\
\sin (90° - A) &= 0.99717.\\
\sin 85°41' &= 0.99716.\\
\text{Tabular difference} &= 3.\\
\tfrac{1}{3} \times 60'' &= 20''.\\
90° - A &= 85°41'20''.
\end{aligned}$$

That is, in this example use the table of sines instead of the table of cosines.

1.9. Angles greater than 90°. The definitions of the six trigonometric functions can be extended to the case of angles greater than 90° (see Fig. 1.4). In Fig. 1.4 the arrow on the x axis

denotes the direction in which x is positive, and the arrow on the y axis denotes the direction in which y is positive. Any angle obtained by rotating counterclockwise from the positive direction of the x axis is a positive angle. When a triangle is constructed for determining the functions of an angle greater than 90° the x axis is always a side of the right triangle (see Fig. 1.5). The hypotenuse is always the line that generates the angle, and the

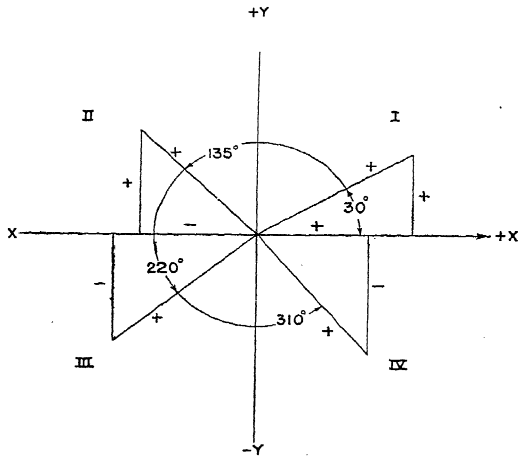


FIG 1.5.

other side of the right triangle is always drawn perpendicular to the x axis. The hypotenuse is always positive. The signs for the sides of the right triangle can be determined by their location with respect to the positive and negative directions of the x axis and y axis.

The four quadrants are numbered with roman numerals in Fig. 1.5. All six trigonometric functions are positive in the first quadrant. The sine and cosecant are positive in the second quadrant, and the other four functions are negative in the second quadrant. The tangent and cotangent are positive in the third quadrant, and the other functions are negative in the third quadrant. The cosine and secant are positive in the fourth quadrant, and the other functions are negative in the fourth quadrant.

The following formulas are useful in finding the trigonometric functions of angles larger than 90°. Here A represents an angle less than 90°.

$\sin(90° \quad A) = \cos A.$		$\sin(90° + A)$	$\cos A.$
$\sin(180° \quad A) = \sin A.$		$\sin(180° + A)$	$-\sin A.$
$\sin(270° \quad A) = -\cos A.$		$\sin(270° + A)$	$-\cos A.$
$\sin(360° \quad A) = -\sin A.$		$\sin(360° + A)$	$\sin A.$
$\cos(90° \quad A) = \sin A.$		$\cos(90° + A)$	$-\sin A.$
$\cos(180° \quad A) = -\cos A.$		$\cos(180° + A)$	$-\cos A.$
$\cos(270° \quad A) = -\sin A.$		$\cos(270° + A)$	$\sin A.$
$\cos(360° \quad A) = \cos A.$		$\cos(360° + A)$	$\cos A.$
$\tan(90° \quad A) = \cot A.$		$\tan(90° + A)$	$-\cot A.$
$\tan(180° \quad A) = -\tan A.$		$\tan(180° + A)$	$\tan A.$
$\tan(270° \quad A) = \cot A.$		$\tan(270° + A)$	$-\cot A.$
$\tan(360° \quad A) = -\tan A.$		$\tan(360° + A)$	$\tan A.$

Example. The cosine of the true angle between two planes is **−0.98717.** Find the angle. If $\cos A = 0.98717$, then $A = 9°11'15''$. Since the cosine is negative, the angle lies in the second quadrant. Therefore the true angle is

$$180° - 9°11'15'' = 170°48'45''.$$

The angle between two planes is always less than 180°, and therefore, when the cosine of the angle between two planes is negative, the angle in the second quadrant should be used as the answer.

Exercises

Using the tables and the formulas for the functions of angles greater than 90°, find the numerical values of the following functions:

1. $\sin 92°16'$.
2. $\cos 93°27'$.
3. $\tan 98°48'$.
4. $\sin 151°10'$.
5. $\cos 171°18'$.
6. $\tan 178°12'$.
7. $\sin 201°23'$.
8. $\cos 224°58'$.
9. $\tan 264°6'$.
10. $\sin 302°58'$.
11. $\cos 315°23'$.
12. $\tan 356°11'$.
13. $\sec 165°15'$.
14. $\csc 201°17'$.
15. $\cot 112°19'$.

1.10. Graphs of the trigonometric functions. The graphs of the trigonometric functions are given in Fig. 1.6. Notice how each function varies as the angle increases from 0° to 360°. From

the graphs the following values of the functions at the given angles may be read:

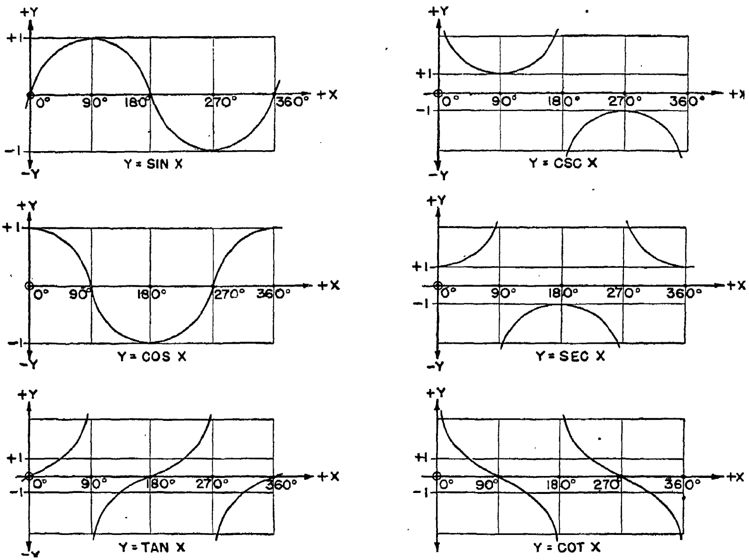


FIG. 1.6.

(*a*)
$$\begin{aligned} \sin 0^\circ &= 0. \\ \sin 90^\circ &= 1. \\ \sin 180^\circ &= 0. \\ \sin 270^\circ &= -1. \\ \sin 360^\circ &= 0. \end{aligned}$$

(*b*)
$$\begin{aligned} \cos 0^\circ &= 1 \\ \cos 90^\circ &= 0 \\ \cos 180^\circ &= -1 \\ \cos 270^\circ &= 0 \\ \cos 360^\circ &= 1 \end{aligned}$$

(*c*)
$$\begin{aligned} \tan 0^\circ &= 0. \\ \tan 90^\circ &= \infty. \\ \tan 180^\circ &= 0. \\ \tan 270^\circ &= \infty. \\ \tan 360^\circ &= 0. \end{aligned}$$

(*d*)
$$\begin{aligned} \sec 0^\circ &= 1 \\ \sec 90^\circ &= \infty \\ \sec 180^\circ &= -1 \\ \sec 270^\circ &= -\infty \\ \sec 360^\circ &= 1 \end{aligned}$$

(*e*)
$$\begin{aligned} \cot 0^\circ &= \infty. \\ \cot 90^\circ &= 0. \\ \cot 180^\circ &= -\infty. \\ \cot 270^\circ &= 0. \\ \cot 360^\circ &= -\infty. \end{aligned}$$

(*f*)
$$\begin{aligned} \csc 0^\circ &= \infty \\ \csc 90^\circ &= 1 \\ \csc 180^\circ &= \infty \\ \csc 270^\circ &= -1 \\ \csc 360^\circ &= -\infty \end{aligned}$$

1.11. Solving oblique triangles when two angles and one side are given. In this case the sine law can be used (see Fig. 1.7).

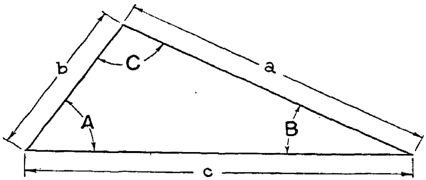


FIG. 1.7

$$\frac{\sin A}{\sin B} = \frac{a}{b}.$$
$$\frac{\sin B}{\sin C} = \frac{b}{c}.$$
$$\frac{\sin A}{\sin C} = \frac{a}{c}.$$

Example. $A = 33°15'$, $C = 65°10'$, $c = 31$. Find a, b, B.

$$\frac{\sin A}{\sin C} = \frac{a}{c}.$$
$$\frac{\sin 33°15'}{\sin 65°10'} = \frac{a}{31}.$$
$$\frac{0.54829}{0.90753} = \frac{a}{31}.$$
$$= 31 \times \frac{0.54829}{0.90753}.$$
$$a = 18.729.$$
$$33°15' + 65°10' = 98°25'.$$
$$180° - 98°25' = 81°35' = B.$$
$$\frac{\sin B}{\sin C} \quad b$$
$$\frac{\sin 81°35'}{\sin 65°10'} \quad \frac{b}{31}.$$
$$0.98923 \quad \frac{b}{31}.$$
$$0.90753$$
$$b = 31 \cdot \frac{0.98923}{0.90753}$$
$$b = 33.791.$$

1.12. Solving oblique triangles when two sides and an angle opposite one of them are given. The sine law can be used in this case.

Example. $A = 130°48'$, $a = 92$, $b = 71$. Find B, C, c.

$$\frac{\sin A}{\sin B} = \frac{a}{b}.$$
$$\frac{\sin 130°48'}{\sin B} = \frac{92}{71}.$$
$$\frac{0.75700}{\sin B} = \frac{92}{71}.$$
$$\sin B = \tfrac{71}{92} \times 0.75700.$$
$$\sin B = 0.58421.$$
$$B = 35°44'50''.$$
$$130°48' + 35°44'50'' = 166°32'50''.$$
$$180° - 166°32'50'' = 13°27'10'' = C.$$
$$\frac{\sin A}{\sin C} = \frac{a}{c}.$$
$$\frac{\sin 130°48'}{\sin 13°27'10''} = \frac{92}{c}.$$
$$\frac{0.75700}{0.23265} \quad 92$$
$$c = 92 \times \frac{0.23265}{0.75700}.$$
$$c = 28.275.$$

If $A < 90°$, $a < b$, $a > b \sin A$, there are two triangles that satisfy the given conditions, and this is therefore called the ambiguous case.

1.13. Solving oblique triangles when two sides and the included angle are given. In this case the cosine law can be used.

$$a^2 = b^2 + c^2 - 2bc \cos A.$$
$$b^2 = a^2 + c^2 - 2ac \cos B.$$
$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Example. $b = 60$, $c = 48$, $A = 41°22'$. Find a, B, C.

$$a^2 = b^2 + c^2 - 2bc \cos A.$$
$$a^2 = 60^2 + 48^2 - 2(60)(48) \cos 41°22'.$$
$$a^2 = 3{,}600 + 2{,}304 - 2(60)(48)(0.75050).$$
$$a^2 = 5{,}904 - 4{,}322.88.$$
$$a^2 = 1{,}581.12.$$
$$a = 39.763.$$

To find B, use

$$\frac{\sin A}{\sin B} = \frac{a}{b}.$$

Then to find C use

$$C = 180° - (A + B).$$

1.14. Solving triangles. The chart given in Fig. 1.8 indicates how right triangles and oblique triangles can be solved.

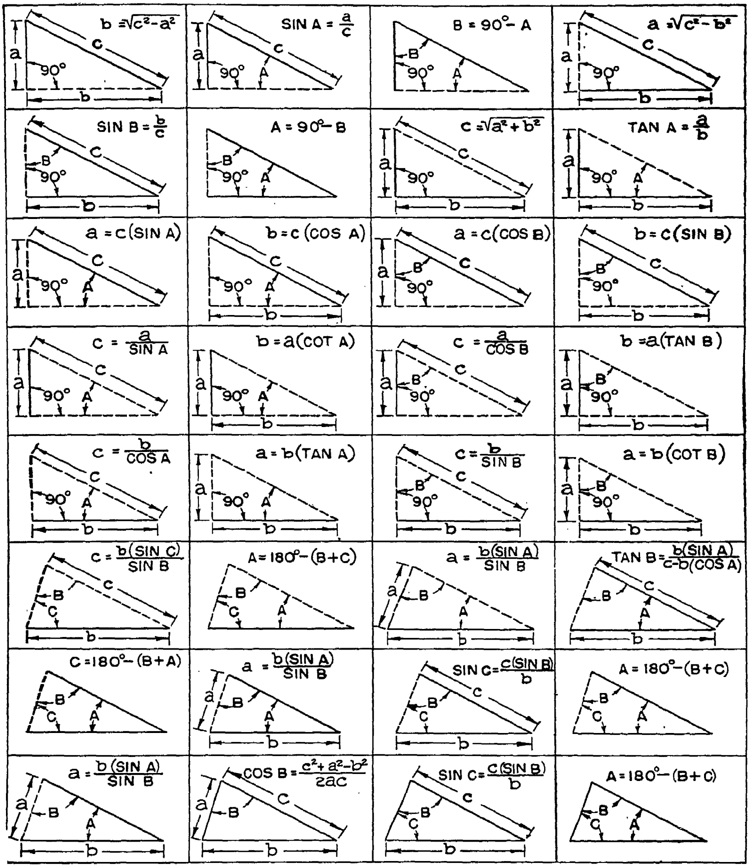


FIG. 1.8.

1.15. Angle revolution chart. In Fig. 1.9 the measured angle α is rotated through the angle β and the true angle is ϕ. If α and β are given, then angle ϕ can be determined from this nomograph.

Example. If α is 21° and β is 48° then the true angle ϕ is determined by laying a straightedge from 21° on the α scale to 48° on the β scale and then reading the result on the ϕ scale, which in this case is 29.7°.

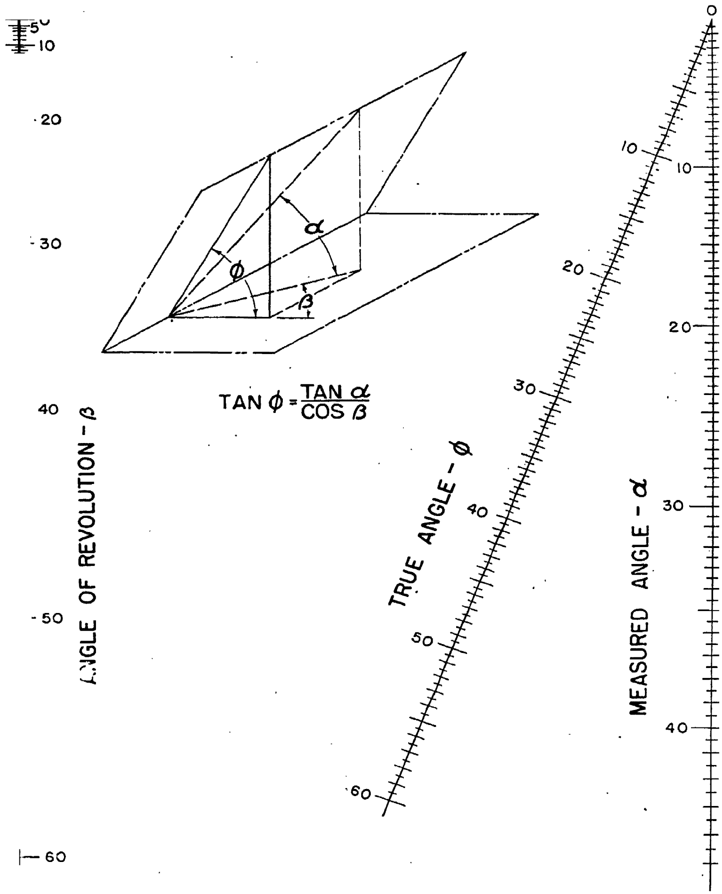


FIG. 1.9.

If α and ϕ are given, then β can be determined by setting the straightedge on the α and ϕ values and reading the answer on the β scale.

If β and ϕ are given, then α can be determined by setting the straightedge on the β and ϕ values and reading the answer on the α scale.

Notice that the plane containing the angle α must be perpendicular to one of the two given planes. It is a common mistake to use this chart when this is not the case. When the plane containing α is not perpendicular to one of the given planes, two rotations are necessary to obtain the true angle. This case will be discussed later. *The angle revolution chart can be used only when the plane containing the measured angle α is perpendicular to one of the given planes.*

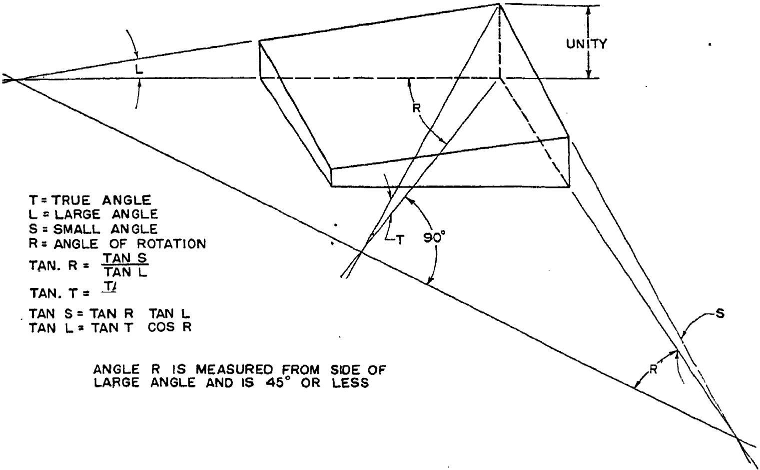


FIG. 1.10.

Exercises

1. $\alpha = 15°$, $\beta = 55°$. Find ϕ.
2. $\alpha = 41°$, $\beta = 32°$. Find ϕ.
3. $\alpha = 36°$, $\beta = 41°30'$. Find ϕ.
4. $\alpha = 9°15'$, $\beta = 52°45'$. Find ϕ.
5. $\beta = 30°$, $\phi = 41°$. Find α.
6. $\beta = 37°$, $\phi \doteq 52°$. Find α.
7. $\alpha = 36°$, $\phi = 43°$, Find β.
8. $\alpha = 39°$, $\phi = 46°$, Find β.

1.16. Milling setup. Figure 1.10 shows a trigonometric method for determining the true angles for a milling setup.

Example. If angle T is 25°10′ and angle R is 32°20′, find angle L and angle S.

$$\begin{aligned}
\tan 25°10' &= 0.46985.\\
\cos 32°20' &= 0.84495.\\
\tan L &= (0.46985)(0.84495).\\
\tan L &= 0.39700.\\
L &= 21°39'11''.\\
\tan S &= (0.63299)(0.39700).\\
\tan S &= 0.25130.\\
S &= 14°6'23''.
\end{aligned}$$

Exercises

1. $S = 15°10'$, $L = 20°30'$. Find R, T.
2. $S = 10°40'$, $L = 25°16'$. Find R, T.
3. $R = 30°15'$, $T = 22°35'$. Find L, S.
4. $R = 31°28'$, $T = 24°17'$. Find L, S.

1.17. Miscellaneous exercises.

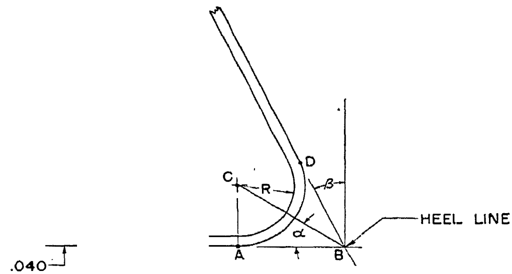


FIG. 1.11.

Example 1. See Fig. 1.11. If the closed bend angle β is 12°, the thickness of the metal is 0.040, and the radius R is 0.165, find the distance AB, from A to the heel line.

$$\begin{aligned}
90° - 12° &= 78°.\\
\alpha &= \tfrac{1}{2} \times 78° = 39°.\\
CA &= 0.165 + 0.040 = 0.205.\\
\frac{AB}{CA} &= \cot \alpha.\\
AB &= CA \cot \alpha.\\
AB &= (0.205)(1.2349).\\
AB &= 0.253.
\end{aligned}$$

Example 2. In Fig. 1.12 the semispan is 500 in. and the root chord is 200 in. If the sweepback angle of the leading edge α is 7° and the sweep-

forward angle of the trailing edge β is 4°32′, find the offsets marked a and b and find the length of the tip chord.

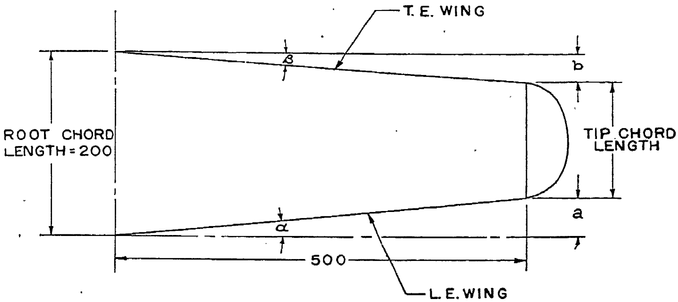


FIG. 1.12.

$$\frac{a}{500} = \tan 7°.$$

$$a = (500)(0.12278).$$

$$a = 61.390.$$

$$\frac{b}{500} = \tan 4°32'.$$

$$b = (500)(0.07929).$$

$$b = 39.645.$$

$$\text{Tip chord} = 200 - (61.390 + 39.645).$$

$$\text{Tip chord} = 98.965.$$

CHAPTER 2

PLANE ANALYTIC GEOMETRY

This chapter constitutes a brief review of certain topics in plane analytic geometry that are essential to the remainder of this book. Since engineering drawings give the projections of points, lines, and planes on a system of three basic reference planes, the material in this chapter is necessary to the study of solid analytic geometry as applied to the airplane. When one

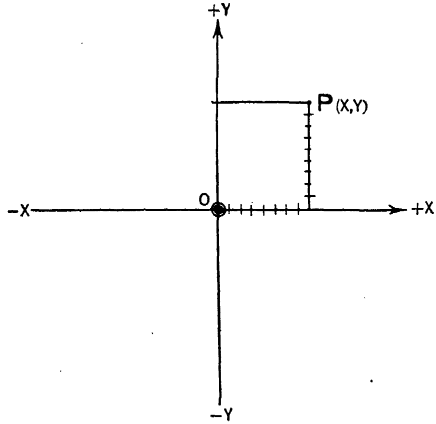


FIG. 2.1.

orthographic view is sufficient to describe an object completely, plane analytic geometry is sufficient to effect the solution of such problems as true length of a line segment, slope of a line, angle between two lines, and distance from a point to a line.

2.1. System of axes. Consider two perpendicular lines intersecting at O. The horizontal line is the x axis, the vertical line is the y axis, and the point O is the origin. A point in a plane is determined by two ordered dimensions, or values, given with reference to the x axis and the y axis. Consider the point P, as shown in Fig. 2.1. The perpendicular distance from P to the

y axis is called the *abscissa* (x distance) of P, and the perpendicular distance from P to the x axis is called the *ordinate* (y distance) of P. The abscissa of P is positive (+) when P lies to the right of the y axis, and is negative (−) when P lies to the left of the y axis. The ordinate of P is positive (+) when P lies above the x axis, and is negative (−) when P lies below the x axis (see Fig. 2.1).

An ordered pair of real numbers determines one point. Thus $(3, -5)$ determines the point which is 3 units to the right of the y axis and 5 units below the x axis. The abscissa is always stated first and the ordinate is always stated second. Every point determines an ordered pair of real numbers.

The x axis and y axis can be thought of as a reference system or reference framework for locating points.

These axes apply to a single plane. In engineering drawings two views are necessary to determine a canted (skew) line or a canted plane. Often three views are given, but two views are *sufficient*, since the third view can be obtained by orthographic projection. The system of axes described above will apply to one view only. For example, the system of axes can be used to act as a reference framework for the plan view of a wing, while another system would be necessary for the front view of the wing. The use of a system of three mutually perpendicular reference axes for three-view drawings will be explained in the next chapter. In this chapter attention will be given to single views, and the projections of points, lines, planes, angles, etc., on a single plane at a time. If, as is sometimes the case, a point, line, or angle lies in the plane of the paper, then this system of axes will be sufficient to describe its location; but, if the object is out of the plane of the paper, then this reference system will describe only its projection on the plane of the paper.

In the plan view of a wing lofted by the wing chord plane method, the chord plane lies in the plane of the paper, and the leading edge of the wing lies in the plane of the paper, so that the equation of the leading edge will completely represent the leading edge. In the case of a wing lofted by the wing reference plane method, the leading edge of the wing is not in the plane of the paper, and the equation of the leading edge in the plan view of the wing will represent only the projection of the leading edge on the wing reference plane.

2.2. Length of a line segment. The length L of the line segment P_1P_2 joining the point $P_1(x_1, y_1)$ to the point $P_2(x_2, y_2)$ is given by the formula

$$L = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

See Fig. 2.2.

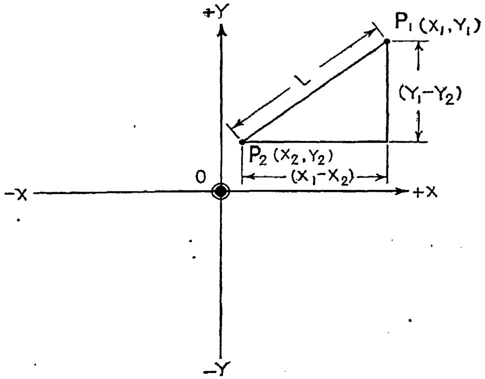


FIG. 2.2.

Example 1. Find the length of the line segment joining the two points (2, 3) and (−4, 7), as shown in Fig. 2.3.

$$L = \sqrt{(-4 - 2)^2 + (7 - 3)^2}.$$
$$L = \sqrt{(-6)^2 + (4)^2}.$$
$$L = \sqrt{36 + 16}.$$
$$L = \sqrt{52}.$$
$$L = 7.211.$$

Example 2. The rear spar trace on the chord plane is a line segment determined by two points. These points are the point of intersection with normal rib station 0, whose coordinates are (0, 23.145), and the point of intersection with normal rib station 135, whose coordinates are (135, 15.346). Find the length of the rear spar trace on the chord plane (see Fig. 2.4).

$$L = \sqrt{(135 - 0)^2 + (15.346 - 23.145)^2}.$$
$$L = \sqrt{(135)^2 + (-7.799)^2}.$$
$$L = 135.225.$$

The formula for the length of a line segment is actually the expression for the length of the hypotenuse of a right triangle, as

obtained by the theorem of Pythagoras (see Fig. 2.2). Notice that the true length of the line segment is not obtained if the line segment is part of a canted line in space. That true length will be discussed in a succeeding chapter. In the present chapter the

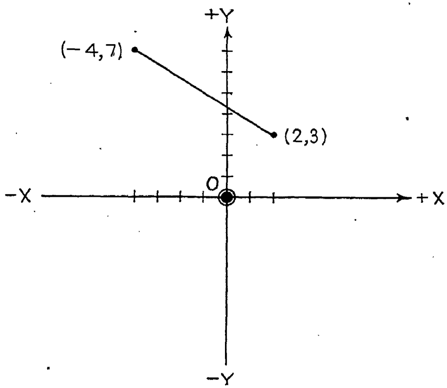


FIG. 2.3.

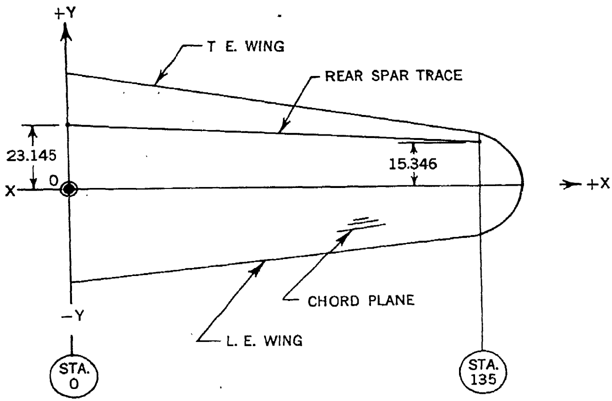


FIG. 2.4.

length obtained is the length of the projection of the canted line segment on the reference plane of the paper. It will be the true length of the line segment if, and only if, one of the other two basic views of the line segment is a level line, in the sense of level lines as used in descriptive geometry and orthographic projection.

These lengths of projected line segments are useful in themselves. In Fig. 2.4 the "rear spar trace" is the intersection of the plane of the forward face of the rear spar with the wing chord plane. In this drawing the wing chord plane is the plane of the paper, and the rear spar is normal (perpendicular) to the plane of the paper. Therefore the length obtained for the rear spar trace is the true length of this segment of the line of intersection, since the trace lies in the plane of the paper.

Exercises

Find the lengths of the line segments joining the following pairs of points:

1. (8, 2) and (4, 1).
2. (25, −3) and (7, 2).
3. (6, −8) and (−17, 4).
4. (−11, 6) and (−3, −9).
5. (−15, −12) and (−25, −16).

2.3. Inclination of a line. The inclination of a line is the angle from the x axis to the line, measured from the x axis to the line

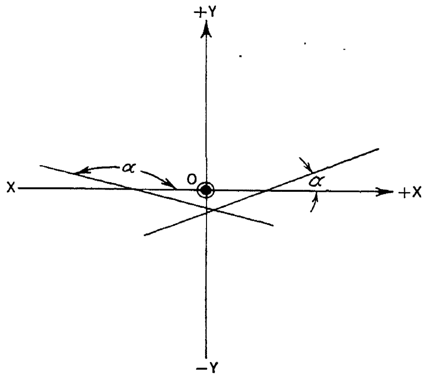


FIG. 2.5.

in a counterclockwise direction (see Fig. 2.5). The inclination is always measured above the x axis.

Example 1. Find the inclination of the line through the two points (10, 3) and (1, 2).

$$\tan \alpha = \frac{3-2}{10-1} = \frac{1}{9} = 0.\dot{1}1111.$$

$$\alpha = 6°20'25''.$$

Example 2. Find the inclination of the line through (6, 17) and (2, 25).

$$\tan \alpha = \frac{17 - 25}{6 - 2} = \frac{-8}{4} = -2.$$
$$\text{arc tan } 2 = 63°26'4''.$$
$$\text{arc tan } -2 = 180° - 63°26'4'' = 116°33'56''.$$
$$\alpha = 116°33'56''.$$

Example 3. Find the inclination of the rear spar trace in Fig. 2.4. This line is determined by the two points (0, 23.145) and (135, 15.346).

$$\tan \alpha = \frac{15.346 - 23.145}{135 - 0}.$$
$$\tan \alpha = \frac{-7.799}{135}.$$
$$\tan \alpha = -0.05777.$$
$$\alpha = 176°41'37''.$$

Notice that when the tangent is negative the angle is in the second quadrant, *i.e.*, the angle lies between 90° and 180°.

This trace, by the definition of the word *trace*, is in the plane of the paper, and the angle is therefore the true angle between the x axis and the trace.

Two lines that have the same inclination are parallel. This applies strictly to two lines which lie in the plane of the paper. If the lines are skew lines in space then it is only their *projections* on the plane of the paper that are parallel, according to this test.

Exercises

Find the inclinations of the lines through the following pairs of points:

1. (17, 23) and (9, 15).
2. (12, −3) and (7, −4).
3. (1, 0) and (16, −9).
4. (−3, −2) and (2, −5).
5. (14, 2) and (16, 9).

2.4. Slope of a line. The slope of a line is the tangent of the angle of inclination. The slope m of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 1. Find the slope of the line through the two points (2, 8) and (1, 3). See Fig. 2.6.

$$m = \frac{8 - 3}{2 - 1} = \frac{5}{1} = 5.$$

Example 2. The slope of the line of intersection of a normal rib plane with the rear spar plane is determined by two points. These points are the point of intersection of the top lofted line of the rear spar on the normal rib plane $P_1(76.149, 10.907)$, and the point of intersection of the bottom lofted

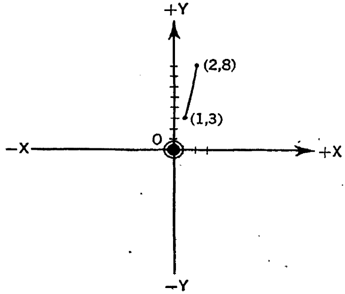


FIG. 2.6.

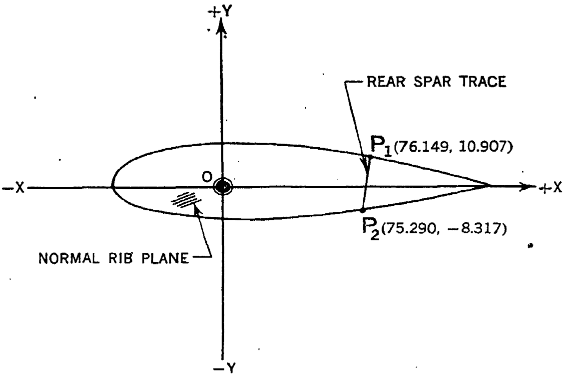


FIG. 2.7.

line of the rear spar on the normal rib plane $P_2(75.290, -8.317)$. See Fig. 2.7.

$$\text{Slope} = m = \frac{10.907 - (-8.317)}{76.149 - 75.290} = \frac{19.224}{0.859} = 22.380.$$

Exercise

Find the slopes of the lines in the exercises in Art. 2.3.

Example 3. Find the slope of the line whose inclination is 2°.

$$\begin{aligned} \alpha &= 2°. \\ m &= \tan \alpha. \\ m &= \tan 2°. \\ m &= 0.03492. \end{aligned}$$

Example 4. Find the inclination of the line of intersection of the plane of a fuselage canted frame with the plane of symmetry. This line is determined by two points: the point of intersection of the canted frame with the top ℄ of airplane (115.375, 56.049) and the point of intersection of the canted frame with the bottom ℄ of airplane (105.437, −40.906).

$$\text{Slope} = m = \frac{56.049 - (-40.906)}{115.375 - 105.437}$$

$$m = \frac{96.955}{9.938} = 9.7560.$$

$$\text{Inclination} = 84°8'51''.$$

Two lines that have the same slope are parallel. If m is the slope of one line and n is the slope of another line, and if $mn = -1$, then the two lines are perpendicular; *i.e.*, two lines are perpendicular if and only if their slopes are negative reciprocals. For example, if the slope of one line is 2 and the slope of another line is $-\frac{1}{2}$, then their slopes are negative reciprocals, $(2)(-\frac{1}{2}) = -1$, and the lines are therefore perpendicular. If the two lines are in the plane of the paper, they are actually parallel or perpendicular, according to these tests, but if they are skew lines in space then their *projections* on the plane of the paper are parallel or perpendicular, according to these tests.

In a subsequent chapter tests will be developed to determine whether two skew lines in space are parallel or perpendicular.

2.5. Point-slope equation of a straight line. If the slope of a given line is m and if the line passes through the point $P_1(x_1, y_1)$, then the equation of the line is

$$y - y_1 = m(x - x_1).$$

This equation can be derived as follows: The slope of a line is $\dfrac{y_2 - y_1}{x_2 - x_1} = m$. Substitute x, y for x_2, y_2. This gives $\dfrac{y - y_1}{x - x_1} = m$, or $y - y_1 = m(x - x_1)$.

Example 1. Find the equation of the line through the point (2, 5) with slope $\frac{1}{3}$.

$$y - 5 = \tfrac{1}{3}(x - 2).$$

Simplifying,

$$3y - 15 = x - 2.$$
$$3y = x + 13.$$
$$y = \tfrac{1}{3}x + \tfrac{13}{3}.$$

When the equation is in this form given values of x may be substituted and the corresponding values of y can be calculated.

Exercises

Find the equations of the lines determined by the following points and slopes:

1. (3, 2) and $m = \frac{1}{2}$.
2. (−4, 6) and $m = 3$.
3. (−5, −7) and $m = -2$.

Reduce the answers to the form $y = mx + b$.

Example 2. Find the equation of the leading edge of the rudder, which is determined as a line intersecting normal rib station 15 at the point 14.196 in. forward of the center line of the hinge, and having a definite "sweepback" angle, the slope of which is −0.12146 (see Fig. 2.8).

$$\begin{aligned} y - y_1 &= m(x - x_1). \\ x_1 &= 15. \\ y_1 &= 14.196. \\ m &= -0.12146. \\ y - 14.196 &= -0.12146(x - 15). \\ y &= -0.12146x + 14.196 + 15(0.12146). \\ y &= -0.12146x + 16.018. \end{aligned}$$

When the equation of the leading edge of the rudder is given in this form, it is possible to determine its point of intersection with any other normal rib plane.

Example 3. Find the coordinates of the point of intersection of normal rib station 70($x = 70$) with the leading edge of the rudder (refer to Fig. 2.8).

$$\begin{aligned} y &= -0.12146x + 16.018 \text{ (refer to Example 2).} \\ x &= 70. \\ y &= (-0.12146)(70) + 16.018. \\ y &= -8.502 + 16.018. \\ y &= 7.516. \end{aligned}$$

Example 4. Find the equation of the trailing edge of the rudder, which is determined as a line intersecting rib station 15 at the point 16.238 aft of the

center line of the hinge, and having a "sweepforward" or slope of 0.18805 (see Fig. 2.8).

$$
\begin{aligned}
y - y_1 &= m(x - x_1). \\
x_1 &= 15. \\
y_1 &= -16.238. \\
m &= 0.18805. \\
y - (-16.238) &= 0.18805(x - 15). \\
y &= 0.18805x - 16.238 - 2.821. \\
y &= 0.18805x - 19.059.
\end{aligned}
$$

Here x represents the station distance as measured along the center line of rudder hinge and y represents the offset aft of the center line. To find the

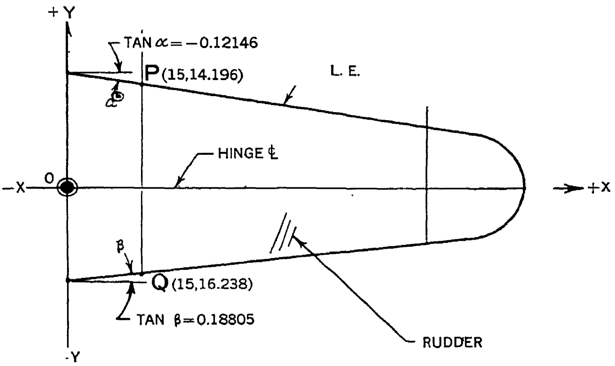


FIG. 2.8.

offset to the trailing edge at station 70, merely substitute $x = 70$ in the equation of the trailing edge.

$$
\begin{aligned}
y &= (0.18805)(70) - 19.059. \\
y &= -5.896.
\end{aligned}
$$

Notice that the equation of a line in this form is well adapted to use with calculating machines. In the equation of the trailing edge, "lock" the number 19.059 in the machine negatively and enter the constant factor 0.18805 on the keyboard. Then any number of x values can be put into the machine and the complete y value can be read directly in one operation. The exact details of this manipulation depend upon the make of the calculating machine, but the principle is the same on most machines.

In writing the equation of a line when the sweepforward or sweepback angle or slope is given, care must be taken to use the correct sign for the value of the slope m (see Fig. 2.5). If the inclination, as defined in Art. 2.5, is less than 90°, the value of m is plus; if it is greater than 90°, the value of m is minus.

In the equation of a line, such as $y = 0.18805x - 19.059$, the quantities 0.18805 and -19.059 are constants and the x and y are variables. The equation of a line can be thought of as a formula that enables one to determine whether or not a given point lies on the line. For example, the point (2, 3) does not lie on the line, because when 2 is substituted for x and 3 is substituted for y the equation is not satisfied, *i.e.*, the left-hand side of the equation is not equal to the right-hand side. However, the point (15, -16.238) does lie on the line, because when 15 is substituted for x and -16.238 is substituted for y the equation is satisfied. Therefore the equation of a line is a relation between pairs of numbers x, y which is satisfied by all those points, and only those points, which lie on the line. In this sense the equation of a line *represents* the line.

2.6. Two-point equation of a straight line. If a given straight line passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ then its equation is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This equation can be derived as follows. The slope of a line is $\frac{\quad - y_1}{x_2 - x_1} = m$. Substitute x, y for x_2, y_2. This gives $\frac{y - y_1}{x - x_1} = m$. Now $\frac{y_2 - y_1}{x_2 - x_1} = m$ and $\frac{y - y_1}{x - x_1} = m$, so $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$. Therefore $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

Example 1. Find the equation of the straight line through $P_1(-1, 6)$ and $P_2(8, 3)$.

$$\begin{aligned}
y - \quad & \quad \frac{3 - \quad}{8 + 1}(x + 1).\\
y - 6 &= -\tfrac{3}{9}(x + 1).\\
9y - 54 &= -3x - 3.\\
9y &= -3x + 51.\\
y &= -\tfrac{1}{3}x + \tfrac{17}{3}.
\end{aligned}$$

Exercises

Find the equations of the straight lines through the following pairs of points:

1. (7, 3) and (6, 8).
2. (−4, 1) and (10, −2).
3. (3, −1) and (0, −4).
4. (−4, −8) and (−7, 2).

Reduce the answers to the form $y = mx + b$.

Example 2. Find the equation of the trailing edge of the elevator. This line is determined as being 30.046 in. aft of the center line of hinge at normal rib station 10, $P_1(10, -30.046)$, and 15.098 in. aft of the hinge center line at rib station 50, $P_2(50, -15.098)$.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

$$x_1 = 10, \qquad y_1 = -30.046.$$

$$x_2 = 50, \qquad y_2 = -15.098.$$

$$y - (-30.046) \quad \frac{-15.098 - (-30.046)}{50 - 10}(x - 10).$$

$$y + 30.046 \quad \frac{14.948}{40}(x - 10).$$

$$y + 30.046 = 0.37370(x - 10).$$

$$y = 0.37370x - 33.783.$$

Notice that the two-point form of the equation of a straight line is actually equivalent to the point-slope form.

$$y - y_1 = m(x - x_1).$$

$$- y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is true because

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

These two forms can be written

$$\frac{- y_1}{x - x_1} = m.$$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The matter of writing the equation of a line is one of the basic fundamentals of this book and is used in later chapters as well as

here. Many major applications of solid analytics to the airplane depend primarily upon writing the equations of lines. Lines in space will be shown to be a simple extension of the instance of a line that lies entirely in one plane.

It is important to check the equation of a line. If a point and slope are given, then the equation should be checked to see whether the given data are satisfied. If two points constitute the given data, the equation should be checked by substituting the coordinates of these two points for x and y in the final equation.

2.7. Slope-intercept equation of a straight line. The equation of the straight line which intersects the y axis at the point $(0, b)$ and has a slope equal to m is

$$y = mx + b.$$

This equation can be derived as follows: The point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$. Let $x_1 = 0$ and $y_1 = b$. Then $y - b = m(x - 0)$, or $y - b = mx$, or finally $y = mx + b$.

Example 1. Find the equation of the straight line which crosses the y axis at the point (0, 3) and which has the slope $\frac{1}{2}$.

$$\begin{aligned} y &= mx + b. \\ b &= 3, \qquad m = \tfrac{1}{2}. \\ y &= \tfrac{1}{2}x + 3. \end{aligned}$$

Example 2. Write the equation of the leading edge of the rudder. This line is determined by its intersection with normal rib station 0, which is 16.018 in. forward of the hinge center line, and its sweepback, or slope, of -0.12146 (see Fig. 2.8).

$$\begin{aligned} y &= mx + b. \\ b &= 16.018. \\ m &= -0.12146. \\ y &= -0.12146x + 16.018. \end{aligned}$$

Notice that the leading edge of the rudder is in the plane of the paper and that this equation therefore completely represents the leading edge.

2.8. Lines parallel to the axes. The equation of a line parallel to the y axis and a units from the origin is $x = a$. In Fig. 2.8 the equation of the station 15 line is $x = 15$ and the equation of the station 70 line is $x = 70$. The equation of a line parallel

to the x axis and a units from the origin is $y = a$. The equation of the rudder hinge center line in Fig. 2.8 is $y = 0$.

2.9. Distance from a point to a line. The distance from the point (x_1, y_1) to the line $y = mx + b$ is

$$D = \frac{mx_1 - y_1 + b}{\sqrt{m^2 + 1}}.$$

This is the length of the perpendicular drawn from the point (x_1, y_1) to the line $y = mx + b$.

Example 1. Find the distance from the point (2, 3) to the line $y = 4x + 5$.

$$\begin{aligned} x_1 &= 2. \\ y_1 &= 3. \\ m &= 4. \\ b &= 5. \\ D &= \frac{(4)(2) - (3) + (5)}{\sqrt{4^2 + 1}} \\ D & \quad \frac{10}{\sqrt{17}} = 2.425. \end{aligned}$$

Example 2. In Fig. 2.8 find the length of the perpendicular from the origin to the leading edge of the rudder. The equation of the leading edge of the rudder is $y = -0.12146x + 16.018$: The origin is the point (0, 0).

$$\begin{aligned} x_1 &= 0. \\ y_1 &= 0. \\ m &= -0.12146. \\ b &= 16.018. \\ D & \quad \frac{(-0.12146)(0) - (0) + (16.018)}{\sqrt{(-0.12146)^2 + 1}} \\ D & \quad \frac{16.018}{1.00735}. \\ D & \quad 15.901. \end{aligned}$$

Example 3. In Fig. 2.8 find the perpendicular distance from the point (5, 8) to the leading edge of the rudder.

$$\begin{aligned} x_1 &= 5. \\ y_1 &= 8. \\ m &= -0.12146. \\ b &= 16.018. \\ D &= \frac{(-0.12146)(5) - (8) + (16.018)}{\sqrt{(-0.12146)^2 + 1}} \\ D &= \frac{7.411}{1.00735} \\ D &= 7.357. \end{aligned}$$

Example 4. In Fig. 2.8, a certain point on station 15 is 5.250 in. from the leading edge of the rudder, the distance from the point to the line being measured normal (perpendicular) to the line. Find the offset of this point from the hinge center line. This offset is the y coordinate of the point.

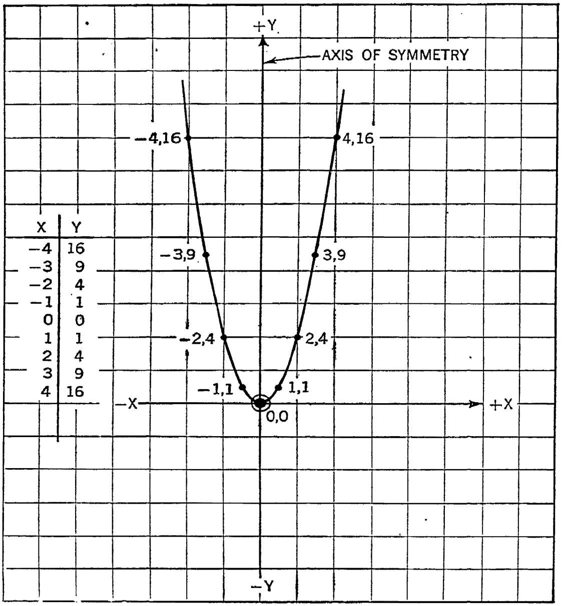


FIG. 2.9.

$$D = \frac{mx_1 - y_1 + b}{\sqrt{m^2 + 1}}$$

$$D = 5.250.$$
$$x_1 = 15.$$
$$m = -0.12146.$$
$$b = 16.018.$$

$$5.250 = \frac{(-0.12146)(15) - y_1 + 16.018}{\sqrt{(-0.12146)^2 + 1}}.$$

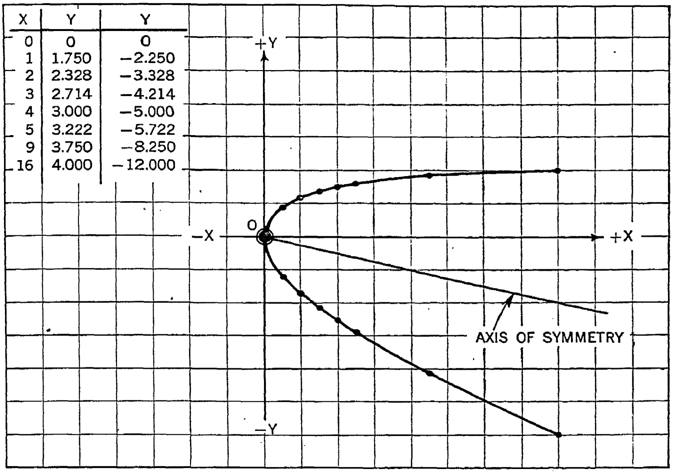
$$5.250 = \frac{-1.822 - y_1 + 16.018}{1.00735}.$$

$$5.250 = \frac{14.196 - y_1}{1.00735}.$$

$$5.289 = 14.196 - y_1.$$

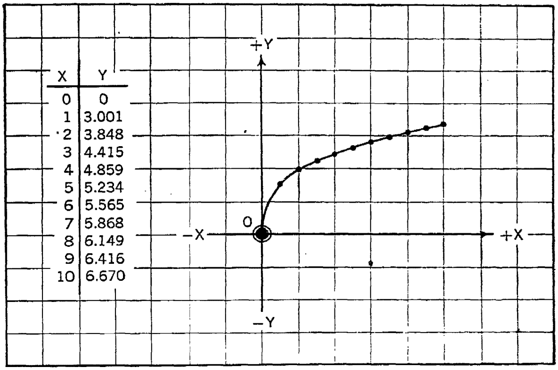
$$y_1 = 8.907.$$

2.10. Plotting curves. Consider the equation $y = x^2$. Assign numerical values to x, and each time calculate the corresponding value of y. This leads to a collection of pairs of numbers, and



X	Y	Y
0	0	0
1	1.750	−2.250
2	2.328	−3.328
3	2.714	−4.214
4	3.000	−5.000
5	3.222	−5.722
9	3.750	−8.250
16	4.000	−12.000

FIG. 2.10.



X	Y
0	0
1	3.001
2	3.848
3	4.415
4	4.859
5	5.234
6	5.565
7	5.868
8	6.149
9	6.416
10	6.670

FIG. 2.11.

each pair of numbers determines a point. The points may be plotted on graph paper, and the collection of points determines a curve. The equation $y = x^2$ represents the curve analytically, and the curve represents the equation geometrically (see Fig. 2.9).

Example 1. Draw the graph of the curve represented by the equation $y = \pm 2\sqrt{x} - \frac{1}{4}x$ (see Fig. 2.10). This is a curve of a type which is very useful in design and lofting. It will be discussed in detail in a subsequent chapter.

Example 2 (see Fig. 2.11). Draw the graph of the curve

$$y = -1.250x + \sqrt{2.075x^2 + 16x}.$$

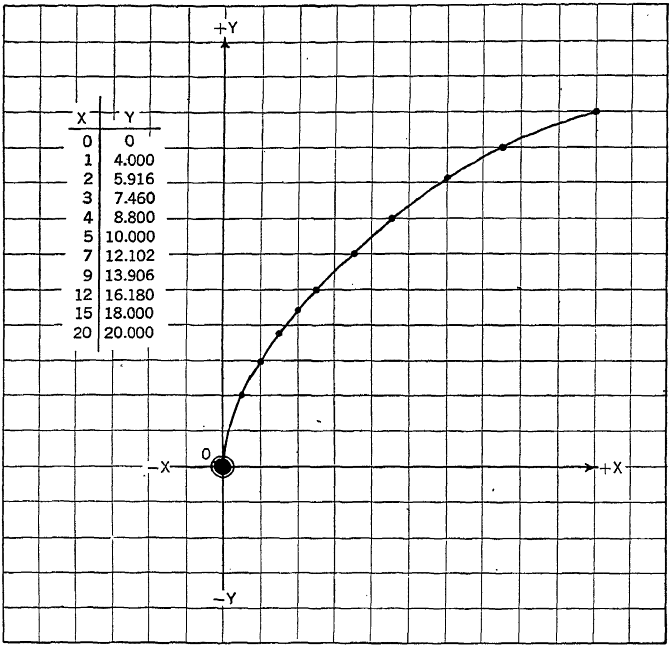


FIG. 2.12.

Example 3 (see Fig. 2.12). Draw the graph of the curve

$$y = 0.6x + \sqrt{-0.44x^2 + 12x}.$$

Exercises

Draw the graphs of the following curves:

1. $y = 3x^2$.

2. $y = -1.5x + \sqrt{2x^2 + 15x}$.

3. $y = 0.5x + \sqrt{-0.5x^2 + 10x}$.

CHAPTER 3

CARTESIAN COORDINATES

In plane analytic geometry points are located with reference to a pair of perpendicular lines. After this reference system is established many relations can be developed. In Chap. 2 we examined some of these relations, such as the distance between two points, the inclination and slope of a line, the distance from a point to a line, equations of straight lines and curves, and many others. This type of reference system is adapted to relations in a single plane or to projections of points, lines, angles, and curves on a plane.

In solid analytic geometry a reference system based on three mutually perpendicular lines is used to develop a corresponding body of theory dealing with points, straight lines, planes, angles, and curves in space. Since engineering drawings deal largely with objects in space, solid analytic geometry is the natural mathematical tool in working with blueprints and problems that arise in the design, engineering, lofting, and tooling of airplanes. It constitutes an analytical counterpart of descriptive geometry.

Any of the fundamental operations of descriptive geometry can be performed analytically, and the results are as accurate as the given dimensions. Some of these fundamental problems are true length of a line segment, true angle between two lines, true distance from a point to a line, true angle between two planes, true distance from a point to a plane, true angle between a line and a plane, true (shortest) distance between two lines, point of intersection of a line and a plane, line of intersection of two planes, and the true angle between the lines of intersection of two given planes with a third plane. In addition many of the fundamental problems of lofting, such as the development of single- and double-canted ribs and the location of tooling points and jigging angles, can be treated by these analytical methods.

The application of solid analytic geometry to problems ordinarily solved by descriptive geometry layout methods is one

of the chief objectives of this book. The purpose is not to supplant layout methods entirely, but rather to supplement layouts and keep the layouts mathematically accurate.

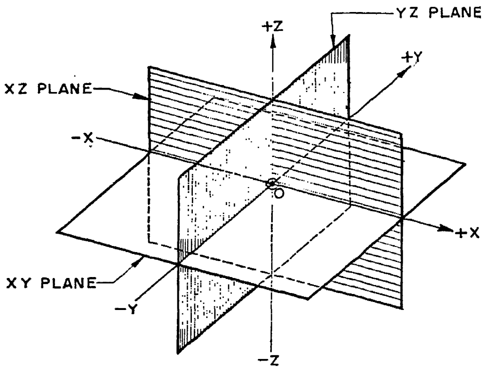


FIG. 3.1.

3.1. Coordinates in space. The idea of the x axis, y axis, and origin system of locating points in a plane can be extended to three dimensions in space, as follows:

Consider three mutually perpendicular planes xy, yz, and xz intersecting at O, the origin. The planes xz and xy intersect in the x axis, the planes xy and yz intersect in the y axis, and the planes yz and xz intersect in the z axis (see Fig. 3.1). The positive and negative directions along the axes are as indicated in Fig. 3.1. The perpendicular distances from a point P in space to the yz, xz, and xy planes are the x, y, z coordinates of the point P, respectively. To every ordered set of three numbers corresponds a point P, and any point P determines an ordered set of three numbers. The coordinates of a point are always given in the order x, y, z.

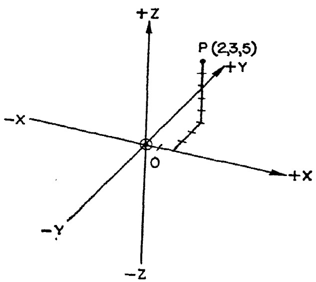


FIG 3.2.

To locate the point P whose coordinates are (2, 3, 5), measure 2 units along the x axis in the positive direction, then measure 3 units parallel to the y axis in the positive direction, and measure

5 units parallel to the z axis in the positive direction. We arrive at the point (2, 3, 5). See Fig. 3.2. For three orthographic views of the axes, see Fig. 3.3.

3.2. Cartesian coordinates in the airplane. The establishment of a reference system for the airplane is the first step in applying the concepts and methods of solid analytic geometry to the airplane. The location of the reference system of axes for the airplane in rigged (flying) position is determined first and depends upon the design of the particular airplane under consideration.

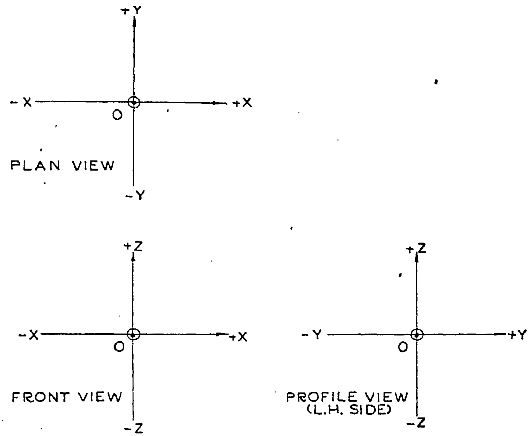


FIG. 3.3.

The choice of location depends upon the design of the wing and the relation of the wing to the fuselage. Wings are usually designed according to the chord plane system, the wing reference plane system, or combinations of these two systems. We shall restrict our attention to a typical example of the chord plane system and a typical example of the wing reference plane system. Since the airplane is in almost all cases geometrically symmetrical about the plane of symmetry of the fuselage, we shall deal with the left-hand side of the airplane unless otherwise specified.

3.3. Rigged axes in a chord plane wing. In a chord plane wing the leading edge, trailing edge, root chord, and tip chord lie in one plane (see Fig. 3.4). This wing is attached to the fuselage by rotating the wing through an angle of incidence and

an angle of dihedral. In this case the rigged (flying) position system of axes can be conveniently located as follows.

Let the yz plane be the plane of symmetry. The plane of symmetry is a vertical fore-and-aft *plane* that "slices" the air-

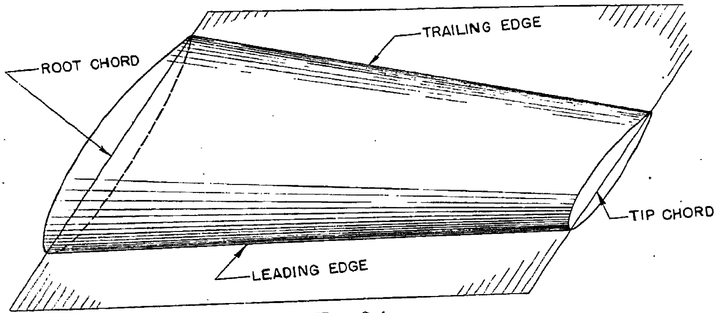


FIG. 3.4.

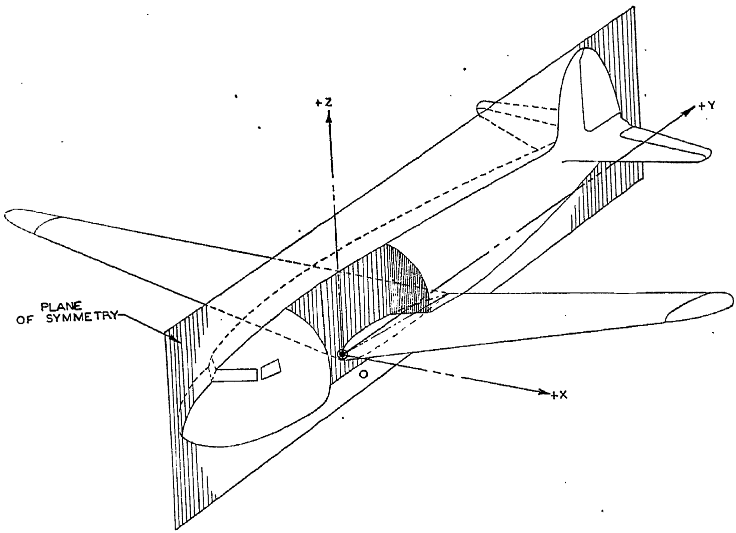


FIG. 3.5.

plane in half. It is sometimes called *center plane* of airplane or center line of ship. Let the xy plane be a horizontal plane, parallel to the ground when the airplane is in rigged position.

Let the xz plane be a vertical plane, perpendicular to the xy and yz planes. Now the exact location of the origin is dictated by the design of the airplane. In our case we shall take it to be the point where the leading edge of the wing intersects the plane of symmetry. In some airplanes it might be convenient to select the point where the trailing edge of the wing intersects the plane of symmetry. The positive direction of the z axis in our case will be the upward direction, and the positive direction of the x axis will be outboard away from the plane of symmetry. The

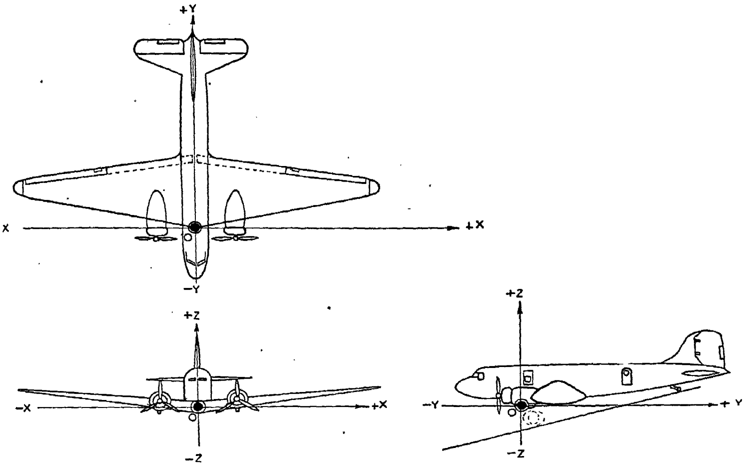


FIG. 3.6.

positive direction of the y axis might be taken either forward or aft. We shall take it in the aft direction (see Fig. 3.5).

This reference system of axes constitutes the rigged axes and is the basic system of axes. By means of it any point, line, or plane in the airplane can be located conveniently. Setting up this reference system for a particular airplane makes available all the concepts and methods of solid analytic geometry as a mathematical tool in solving problems.

For three orthographic views of the rigged axes, see Fig. 3.6.

Notice that the positive direction of the x axis is outboard on the left-hand side of the airplane, the positive direction of the y axis is aft, and the positive direction of the z axis is upward.

3.4. Rigged axes in a wing reference plane wing. In a wing reference plane wing, the chord plane may be flat, or it may be twisted. If it is twisted, the leading edge, trailing edge, root chord, and tip chord do not lie in a single plane. They constitute, instead, a skew quadrilateral in space. This is sometimes called a wing with aerodynamic twist. In this text we shall always deal with a wing reference plane wing with aerodynamic twist, with

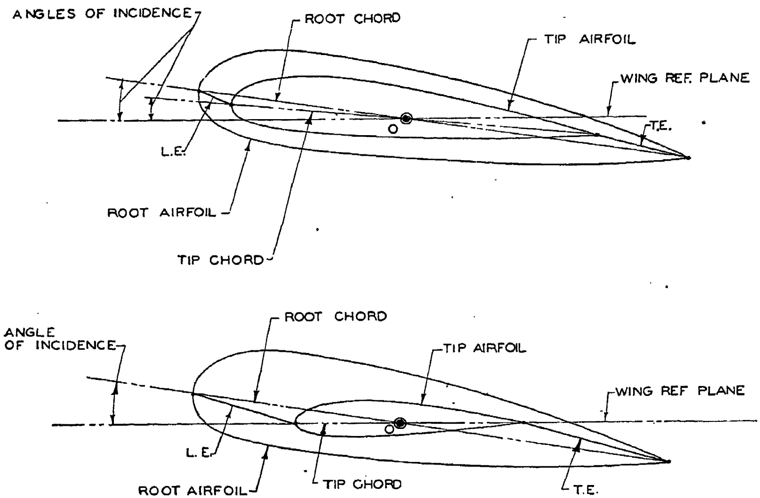


FIG. 3.7.

the angle of incidence as measured from the wing reference plane being less at the tip than at the root airfoil of the wing (see Fig. 3.7).

In the case of a wing with aerodynamic twist it is customary to use the intersection line of the wing reference plane and the common per cent plane as the axis of twist. Therefore it is usually most convenient to take the origin for the rigged axes at the point where this line intersects the plane of symmetry and establish a set of basic xy, xz, and yz reference planes in the same manner as that in Art. 3.3 (see Fig. 3.8). This wing is established in relation to these basic planes by rotating the wing reference plane ($x_w y_w$ plane) through an angle of dihedral only.

3.5. Systems of axes in general. In the two previous articles we described two typical types of systems of basic axes. The airplane can now be divided into four major parts: fuselage,

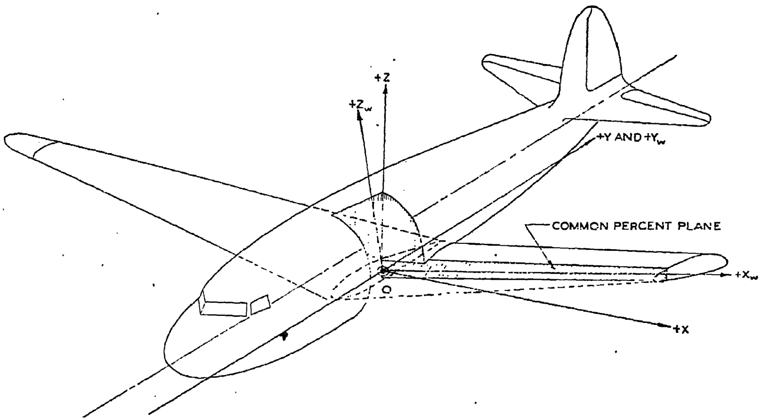


FIG. 3.8.

wing, nacelles and engine sections, and empennage. These in turn can be subdivided as follows:

1. Fuselage.
   a. Forebody.
   b. Center section or main body.
   c. Afterbody.
2. Wing.
   a. Inner wing.
   b. Outer wing.
   c. Wing control surfaces.
3. Nacelles and engine sections (depending on the number of engines).
4. Empennage.
   a. Vertical stabilizer.
   b. Rudder.
   c. Horizontal stabilizer.
   d. Elevator.

The above breakdown is typical of many airplanes. Naturally, the design of the particular airplane under consideration determines the exact nature of the breakdown. The above is typical enough to serve as a concrete illustration.

As the production is increased on a specific model these major divisions are subdivided into many more minor assemblies in order to facilitate production.

Each subassembly is determined mathematically by an origin and three coordinate axes. The rigged axes are basic and fundamental, but there must also be a set of axes for the wing, another for the empennage, another for the nacelles, etc.

3.6. Chord plane wing axes. If the wing is lofted by the chord plane system the wing is attached to the fuselage by rotating

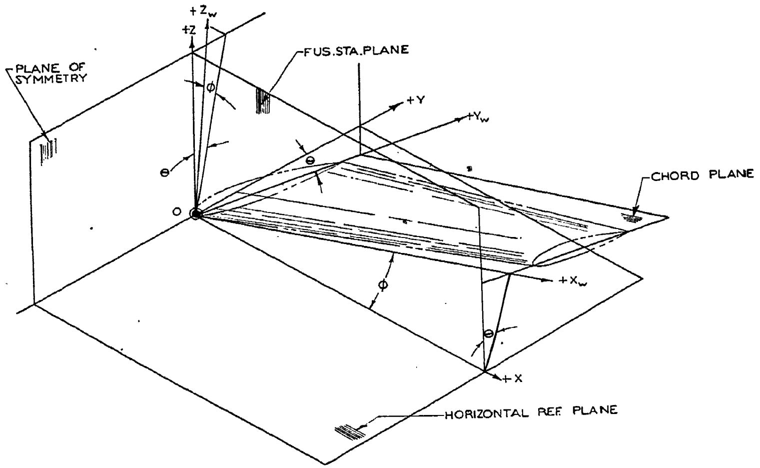


FIG. 3.9.

through two angles: an angle of incidence and an angle of dihedral. The angles of incidence and dihedral as used here are shown in Fig. 3.9.

If a point in the wing is located with respect to the chord plane system of axes, it can be located with reference to the rigged system of axes by means of formulas that will be developed in a subsequent chapter. A set of similar formulas will also be given to locate points with reference to the chord plane system of axes when the rigged coordinates are given. Such formulas are of particular value in landing gear problems, for example, where the engineering drawings involve both chord plane and rigged dimensions.

We use subscripts to denote rotation in order to distinguish the wing axes from the rigged axes. For an orthographic drawing of a chord plane wing in the loft layout view, see Fig. 3.10.

Notice that the positive direction on the x_w axis is outboard, the positive direction on the y_w axis is aft, and the positive direction on the z_w axis is upward.

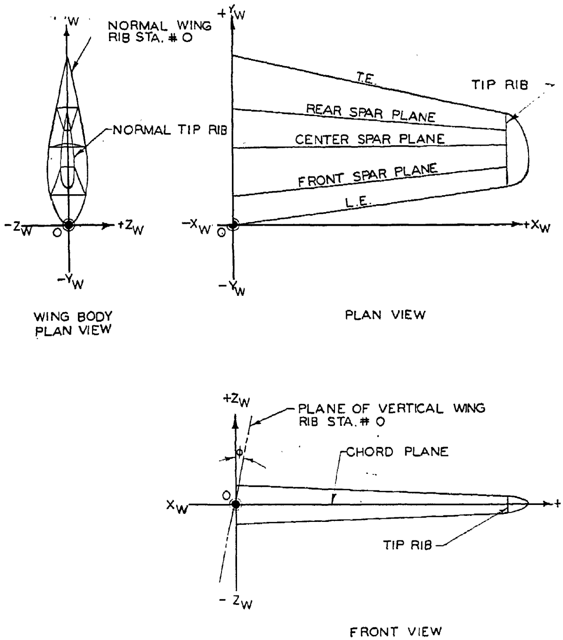


FIG. 3.10.

In Fig. 3.10, the x_w axis, y_w axis, and z_w axis are all in the plane of the paper. In the plan view the chord plane, as determined by the root chord, tip chord, leading edge, and trailing edge, is in the plane of the paper. In the front view the chord plane is seen on edge, normal to the plane of the paper. In the end view (wing body plan view) the normal tip rib and the normal wing rib at station zero are in the plane of the paper. In the front view the

vertical ribs are normal to the plane of the paper and are canted at an angle ϕ to the z_w axis, where ϕ is the angle of dihedral.

Vertical ribs are parallel to the plane of symmetry, and normal ribs are perpendicular to the chord plane. The spars are usually designed as in Fig. 3.10, normal to the chord plane.

For an orthographic drawing of a chord plane wing as seen in the rigged position, see Fig. 3.11.

In Fig. 3.11 the leading edge of the wing is a canted line, *i.e.*, its true length is not shown in any of the three views. The

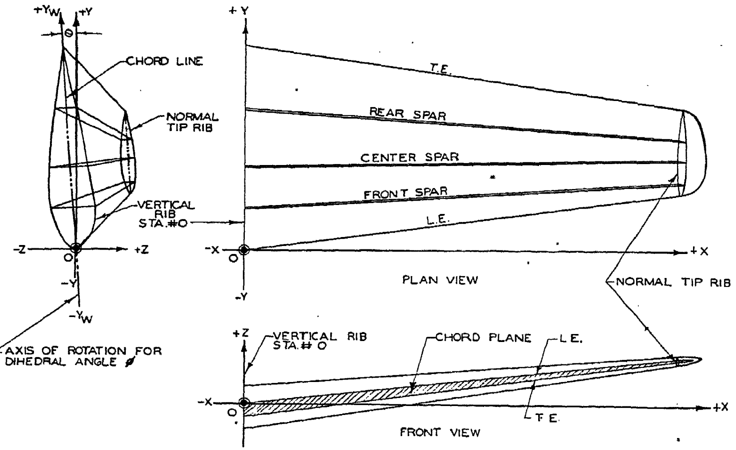


FIG. 3.11.—Wing in chord plane system, rigged view.

vertical ribs are perpendicular to the x axis. This x axis must not be confused with the x_w axis of Fig. 3.10. The x axis, y axis, and z axis of Fig. 3.11 are rigged axes, as described in Art. 3.2. The normal ribs are canted in the views shown in Fig. 3.11.

3.7. Incidence and dihedral. The terms *incidence* and *dihedral* are common terms in the aircraft industry, but it is usually necessary to define them precisely as they are intended to be interpreted. Refer to Fig. 3.9. The angle of incidence is measured in the plane of symmetry. It might be helpful to think of the positive y axis being rotated downward from the y position to the y_w position, the rotation being about the x axis. The y axis and the y_w axis are in the plane of symmetry. This

rotation introduces the angle θ, which is the angle of incidence. Next sight into the positive end of the y_w axis, and rotate the x axis and z axis about the y_w axis so as to introduce dihedral, thus establishing the x_w axis and the z_w axis (see Figs. 3.9 and 3.11). Remember that the x, y, z axes are the rigged axes, for the airplane in flying position, and the x_w, y_w, z_w axes are the wing axes obtained after rotating the original x, y, z set for incidence and dihedral.

3.8. Wing reference plane axes. When a wing is designed with aerodynamic twist, the incidence angle varies from root to

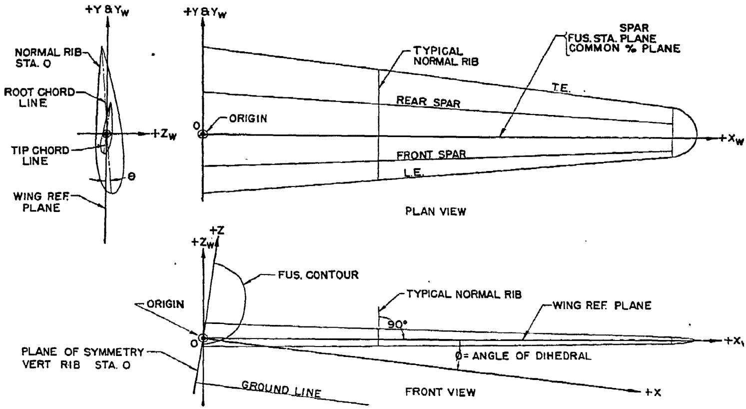


FIG. 3.12.—Wing in reference plane system.

tip. There is no chord plane because the leading edge, trailing edge, root chord, and tip chord form a warped quadrilateral. It is customary to set up a wing reference plane that is used as a reference plane. The x_w axis is usually taken as the common per cent line in the plan view (see Fig. 3.12). In Fig. 3.12 the wing reference plane is in the plane of the paper in the plan view and is on the edge, normal to the plane of the paper in the front view. In the front view the x axis, x_w axis, z axis, and z_w axis are all in the plane of the paper. The angle of dihedral ϕ is therefore true size as seen in the front view.

The x, y, z system of axes can be rotated into the x_w, y_w, z_w system by rotation through dihedral only. The rotation takes place about the y axis as an axis of rotation.

When a point is located with reference to the x, y, z axes it can be determined with reference to the x_w, y_w, z_w axes, and vice versa, by means of a set of formulas which will be developed in a subsequent chapter. Since only one angle of rotation is involved, the formulas are simpler than those for the wing chord plane system, where two angles of rotation are involved.

The origin is determined by the intersection of three mutually perpendicular planes: the normal rib plane at wing station 0, the wing reference plane, and the common per cent plane. The common per cent plane is perpendicular to the normal rib plane at station 0 and intersects each normal rib at the same per cent of its total length measured from either the leading edge or trailing edge of the wing.

The normal rib plane at station 0 is the y_wz_w plane, and all normal ribs are parallel to this plane. The wing reference plane is the x_wy_w plane, which is a plane perpendicular to a normal rib plane. It makes an angle θ with the root chord in the plane of the normal rib at station 0. The common per cent plane is the x_wz_w plane, which is perpendicular to both the x_wy_w plane and the y_wz_w plane.

All measurements are taken from the wing reference plane in the lofted position of the wing, and therefore only one angle of rotation, that of dihedral, is necessary to put the wing into rigged (flying) position. The positive direction on the x_w axis is outboard, the positive direction on the y_w axis is aft, and the positive direction on the z_w axis is upward.

3.9. Special wings. Some wings are designed so as to have incidence only from the plane of symmetry to a certain point outboard, and from that point to the tip the wing has both incidence and dihedral. Such a wing would need two sets of wing axes.

3.10. System of axes for nacelle position. As usual, the location of the axes depends upon the design of the nacelle. In nacelle position the origin is usually located by the intersection of three planes:

1. The vertical thrust plane.
2. The nacelle station 0 plane.
3. The horizontal thrust plane.

The intersection of the vertical thrust plane and the horizontal thrust plane determines the center line of thrust of the engine

(see Fig. 3.13). The vertical thrust plane is the y_nz_n plane, the nacelle station 0 plane is the x_nz_n plane, and the horizontal thrust plane is the x_ny_n plane. The subscript n identifies the axes as nacelle axes.

The x_n axis is the line of intersection of the horizontal thrust plane and the nacelle station 0 plane, *i.e.*, it is the line of intersection of the x_ny_n and x_nz_n planes. The x_n axis is perpendicular to the vertical thrust plane, *i.e.*, it is perpendicular to the y_nz_n

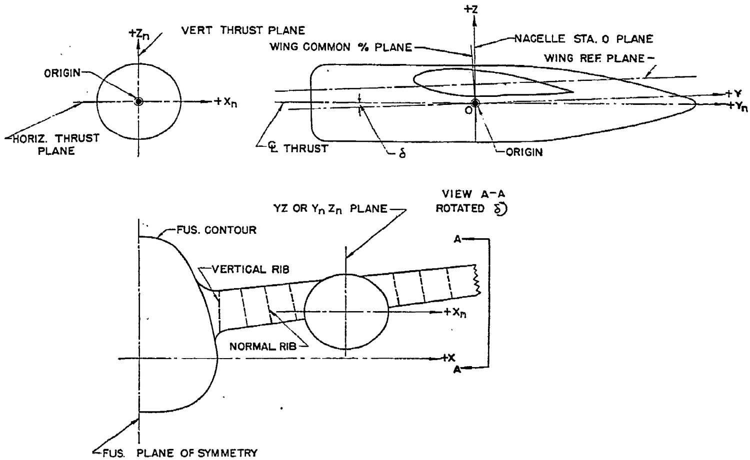


FIG. 3.13.

plane. The positive half of the x_n axis is outboard from the vertical thrust plane, and the negative half of the x_n axis is inboard from this plane.

The y_n axis is the line of intersection of the horizontal thrust plane and the vertical thrust plane, *i.e.*, it is the line of intersection of the x_ny_n and y_nz_n planes. The y_n axis is usually the center line of thrust of the propeller. The y_n axis is perpendicular to nacelle station 0 plane and to all normal nacelle station planes, *i.e.*, it is perpendicular to the x_nz_n plane and to all planes parallel to this plane. The positive direction is aft and the negative direction is forward.

The z_n axis is the line of intersection of the vertical thrust plane and the nacelle station 0 plane, *i.e.*, it is the line of intersection

of the $y_n z_n$ and $x_n z_n$ planes. The z_n axis is perpendicular to the horizontal thrust plane. The positive direction on the z_n axis is upward from the horizontal thrust plane, and the negative direction is downward from this plane.

The x_n axis, y_n axis, and z_n axis occur on the individual engineering drawings as center lines. As soon as a reference system of axes is decided upon, points can be located, and the concepts and methods of solid analytic geometry can be applied to engineering drawings dealing with the engine nacelles.

In Fig. 3.13 the nacelle is shown in relation to the wing. Although the relation of the nacelle varies on different airplanes, the design shown here is quite typical. The vertical thrust plane is the same as a vertical rib station plane of the wing. Therefore the x_n axis of the nacelle system of axes and the x axis of the rigged system of axes are parallel. The y_n axis makes an angle, denoted by δ with the y axis, as measured in the vertical thrust plane.

Mathematical formulas will be developed in a subsequent chapter which will enable us to relate points in the nacelle system of axes to the wing system of axes, which in turn we shall be able to relate to the rigged system of axes. Therefore the various subsystems of axes can be related to each other. This is important in tooling, jig building, and other phases of the manufacture of airplanes.

3.11. System of axes for vertical stabilizer and rudder. The vertical stabilizer and rudder are usually lofted in rigged position, *i.e.*, the normal ribs of the vertical stabilizer are parallel to the fuselage reference plane (horizontal reference plane), and the plane of symmetry (chord plane) of the vertical stabilizer and rudder is the same as the plane of symmetry of the fuselage. The axes for the vertical stabilizer and rudder are therefore parallel to those for the rigged position of the airplane, although the origin is not at the same point. The origin is located by the intersection of three planes:

1. The plane of symmetry (chord plane) of the vertical stabilizer.
2. The normal rib plane at station 0.
3. A plane which is determined by the rudder hinge center line and perpendicular to the above two planes.

See Fig. 3.14.

The plane of symmetry (chord plane) of the vertical stabilizer and rudder is the yz plane. The normal rib plane at station 0 is the xy plane. The plane through the rudder hinge center line and perpendicular to the xy plane and yz plane is the xz plane. The xz plane is in most cases the common per cent plane of the vertical stabilizer and rudder (see Fig. 3.14). This illustrates a typical method of design. Other designs, such as a twin-rudder design, or a single-engine airplane with a single

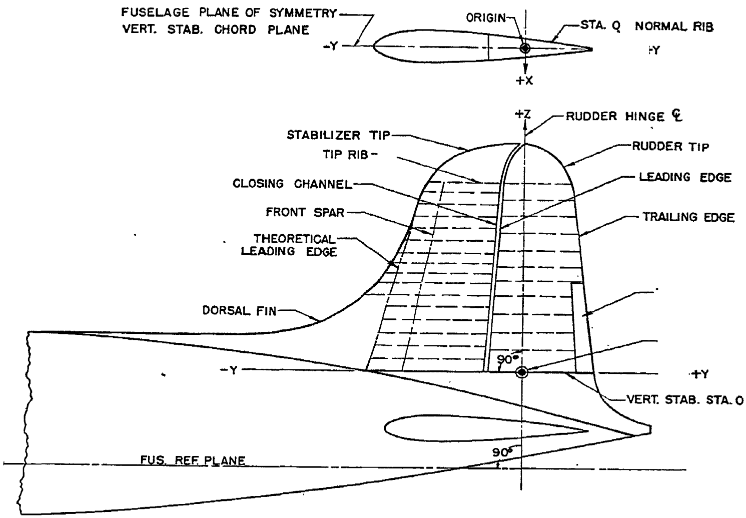


FIG. 3.14.

propeller, where incidence is put into the vertical surface to eliminate propeller torque, can be assigned a set of axes in keeping with the design.

The x axis is the line of intersection of the xz plane and the normal rib plane at station 0. The x axis is perpendicular to the plane of symmetry (chord plane). The outboard direction on the left-hand side of the airplane is the positive direction of the x axis; the opposite direction is the negative direction.

The y axis is the line of intersection of the normal rib plane at station 0 and the plane of symmetry (chord plane). It is perpendicular to the xz plane. The positive direction is aft

of the xz plane, and the negative direction is forward from that plane.

The z axis is the line of intersection of the plane of symmetry (chord plane) and the xz plane (rudder hinge center line). The z axis is perpendicular to all normal rib planes. The z axis is positive upward from the normal rib plane at station 0. The downward direction is negative.

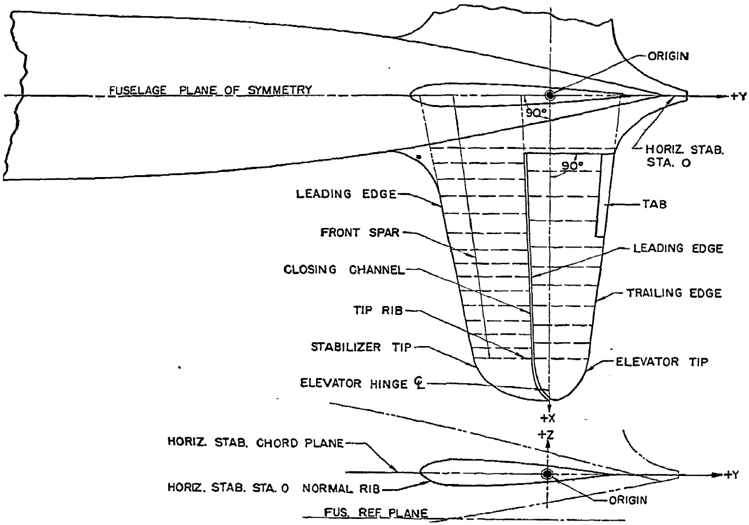


FIG. 3.15.

3.12. System of axes for horizontal stabilizer and elevator. Usually the design of the horizontal stabilizer is such that it can be treated somewhat like the wing, although it is less complicated mathematically. There are three types of design which are most frequently used, and we shall consider each case in turn.

First, the horizontal stabilizer is sometimes designed so that no rotation whatever is needed to relate its axes to the rigged system of axes, *i.e.*, the axes for the horizontal stabilizer are parallel to the rigged system of axes, but the origin is at a different point. In this case the rigged view and lofted view are the same.

We denote the axes by x, y, z (see Fig. 3.15). The origin is located as the point of intersection of three planes:

1. The normal rib plane at station 0.
2. The plane of symmetry of the horizontal stabilizer and elevator.
3. The plane that is determined by the elevator center line of hinge and is perpendicular to the above two planes.

The normal rib plane at station 0 is the yz plane. The plane of symmetry of the horizontal stabilizer and elevator is the xy plane. The plane that is determined by the elevator center line of hinge and is perpendicular to the yz and xy planes is the xz plane. The xz plane is in most cases the common per cent plane of the horizontal stabilizer and elevator. The x axis is the line of intersection of the plane of symmetry and the xz plane and is perpendicular to all normal rib planes. The positive direction of the x axis is outboard on the left-hand side of the airplane, and the negative direction is opposite to this. The y axis is the line of intersection of the normal rib plane at station 0 and the plane of symmetry and is perpendicular to the xz plane. The positive direction of the y axis is aft, and the negative direction is forward. The z axis is the line of intersection of the normal rib plane at station 0 and the xz plane and is perpendicular to the plane of symmetry. The positive direction of the z axis is upward and the negative direction is downward.

Second, there may be only one angle of rotation necessary to rotate the horizontal stabilizer and elevator from their lofted position to their rigged position. This is the angle of incidence, denoted by θ. In this case the lofted position of the system of axes for the horizontal stabilizer and elevator may be denoted by x_h, y_h, z_h. Figure 3.16 shows the horizontal stabilizer and elevator rigged with an angle of incidence only. The angle of incidence is measured in the plane of the normal rib at station 0. This is the same as the plane of symmetry of the airplane. The x_h and x axes are parallel. The y_h and y axes make an angle θ with each other. The z_h and z axes make an angle θ with each other. Sometimes an angle of dihedral ϕ instead of an angle of incidence θ is used, but the two cases are quite similar mathematically.

Third, there may be two angles of rotation necessary to rotate the horizontal stabilizer and elevator from the lofted position to the rigged position. These angles are an angle of incidence and

an angle of dihedral. This case is not very common, but it can be treated in the same way as a chord plane wing.

3.13. System of axes for the fuselage. The fuselage is relatively simple because it is usually lofted in its rigged position. In fact the rigged system of axes can be used to serve as the fuselage reference system. It is usually convenient to do this, because the wing can be related directly to the fuselage and the empennage can be related directly to the fuselage. In the

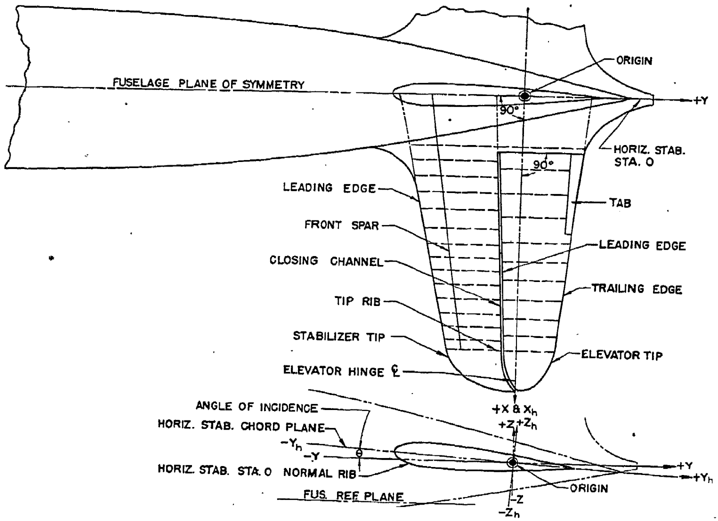


FIG. 3.16.

rigged system (fuselage reference system) the x coordinates are buttock lines, the y coordinates are fuselage station lines, and the z coordinates are water lines. If buttock line 0 (plane) is taken to be the plane of symmetry of the airplane, as it usually is, then the point (12, 50, 13) would be on buttock line 12. If the fuselage station 0 (plane) is taken perpendicular to the plane of symmetry at the origin of the rigged system of axes, then the point (12, 50, 13) would lie on fuselage station 50. If the water line 0 (plane) is taken as the xy plane of the rigged system of axes then the point (12, 50, 13) is on water line 13.

In other words, it is possible to select buttock line 0, fuselage station 0, and water line 0 at the origin of the rigged system, and

if this is done, then the x, y, z coordinates of a point are exactly the buttock line, fuselage station line, and water line, respectively, of the point. If the buttock line 0, fuselage station 0, and water line 0 are established first, then it would be convenient to place the origin of the rigged system of axes (fuselage reference system) at the intersection of these three planes, and then the x, y, z coordinates of a point would be exactly the buttock line, fuselage station line, and water line of the point.

Since the terms *buttock line*, *station line*, and *water line* have been carried over from shipyard nomenclature and have little special meaning on airplanes, it would be convenient to drop this terminology altogether and refer instead to the x, y, z coordinates of a point and the xy, yz, xz planes.

Many more subassembly systems of axes could be cited here, but the ones mentioned are typical and others can be set up in a similar way.

It is important, in the case of any particular airplane, to set up the basic rigged, wing, nacelle, and empennage systems of axes and to locate them in convenient ways, depending upon the design of the airplane. It is useful to make drawings, such as the ones given in this chapter, to show the exact locations of the various systems of axes. It is then necessary to relate the various systems of axes to each other by means of equations of translation and rotation, which will be explained in a subsequent chapter.

CHAPTER 4

TRUE LENGTHS AND TRUE ANGLES

In Chap. 3 we described the idea of using a set of three mutually perpendicular lines in space to locate points. A line is determined by two points. In this chapter we shall define and show how to calculate the direction ratios of a line and the direction cosines of a line. By means of direction ratios and direction cosines we can develop formulas to find the true angle between two lines and the true angle between a line and any one of the three basic reference planes. We shall also learn how to find the true length of a line segment. The methods developed in this chapter constitute the foundation for most of the applications of solid analytic geometry to the airplane.

Most of our applications will be to canted lines in space. If a line or line segment lies in the plane of the paper it can be represented graphically by one view only. Therefore ordinary plane trigonometry or plane analytic geometry will suffice to deal with it mathematically. However, we shall be interested mainly in space problems, *i.e.*, in lines and line segments that require two views to determine them. It is here that the value of solid analytic geometry will be clearly demonstrated.

For example, it is possible to calculate the true length of a line segment by imagining the line segment to be a diagonal of a "box," or rectangular parallelepiped, and then solving two certain right triangles in succession, using trigonometry or the theorem of Pythagoras. Our method reduces the calculation to a simple formula, and this is typical of other kinds of applications.

Another example is the problem of finding the true angle between two skew lines in space. This can be solved by making a descriptive geometry layout and determining the angle graphically. It can also be solved by trigonometry, but the method of solution is quite complicated and requires a relatively new attack on each problem that arises. *By using the methods of solid analytic geometry we reduce the solution of all problems on true*

angle between two lines to the same simple routine, and the result can be obtained to a degree of accuracy that is limited only by the accuracy of the given data.

4.1. Direction ratios. As explained in Chap. 3, a point can be located by three ordered numbers. Two points determine a line. Consider the line determined by the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. The direction ratios of this line are defined to be

$$x_2 - x_1 : y_2 - y_1 : z_2 - z_1 \qquad \text{or} \qquad x_1 - x_2 : y_1 - y_2 : z_1 - z_2.$$

In the case of the line through the two points $A(4, 5, 6)$ $B(8, 7, 3)$ the direction ratios are

$$4:2:-3 \qquad \text{or} \qquad -4:-2:3.$$

The direction ratios of a line determine the direction of the line but do not determine the exact location of the line in space. If the direction ratios of a certain line are $a:b:c$, then the numbers $ka:kb:kc$ are also direction ratios of the line, where k is any number different from zero, *i.e.*, the direction ratios can be divided or multiplied by any number. For purposes of calculation, it is often convenient to divide by a number that will reduce one of the three numbers to unity. For example, the direction ratios $4:2:-3$ can be divided by 4, 2, or -3, respectively, to give

$$1:0.5:-0.75$$
$$2:1:-1.5$$
$$-1.33:-0.67:1.$$

Example. The front spar top lofted* line is determined by two points: its point of intersection on normal rib station 0, whose coordinates are (0, -48.091, 10.738), and its point of intersection on normal rib station 152, whose coordinates are (152, -20.040, 6.435). Find its direction ratios.

$$152 - 0 : -20.040 + 48.091 : 6.435 - 10.738$$
$$152 : 28.051 : -4.303$$

It is often best to divide through by the largest of the three direction ratios. Dividing by 152,

$$1:0.18455:-0.02831.$$

* The term *mold line* is often used here, but requires a careful definition. The mold line is usually defined to be inside the skin (outer covering) of the airplane. With the advent of laminar flow airfoils and smoother outside surfaces it becomes necessary to work to the outer contour of the surface. In this case the term *lofted line* is more appropriate than *mold line*, the lofted line usually being located on the outer contour in such instances.

Here 0.18455 is the tangent of the angle between the x axis and the projection of the front spar top lofted line on the xy plane. Also, -0.02831 is the tangent of the angle between the x axis and the projection of the front spar top lofted line on the xz plane. This can be illustrated as shown in Fig. 4.1.

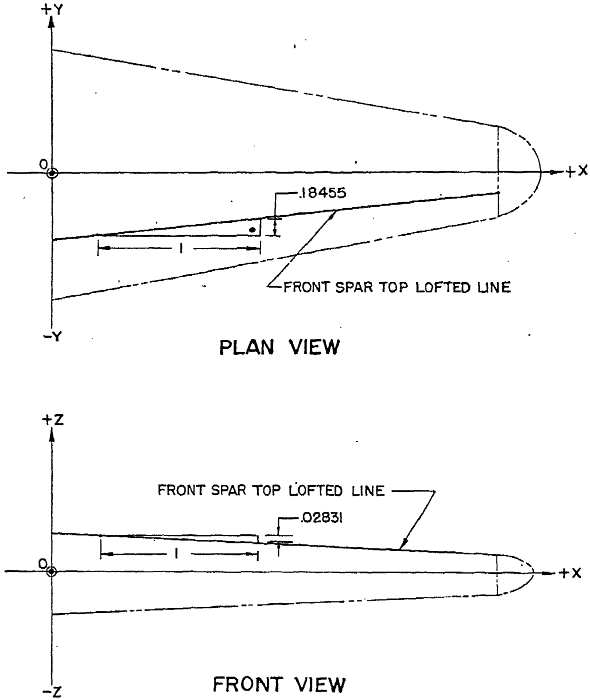


FIG. 4.1.

The direction ratios of a line are the sides of a "box" of which the two points are the extremities of a diagonal. In Fig. 4.2 the direction ratios of the diagonal are $a:b:c$.

Sometimes, to determine the line, two points are not given. Instead, the line may be determined by one point and the projected (apparent) angles in two views. The direction ratios are

$$1:\tan\alpha:\tan\beta.$$

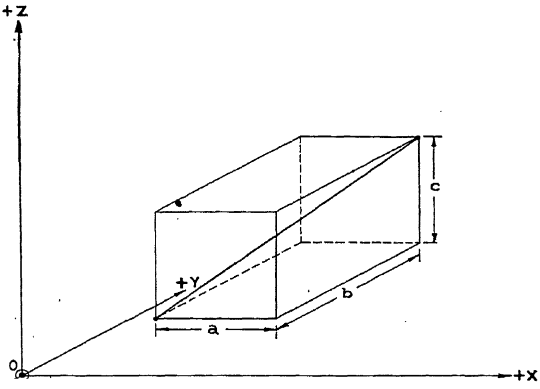


Fig. 4.2.

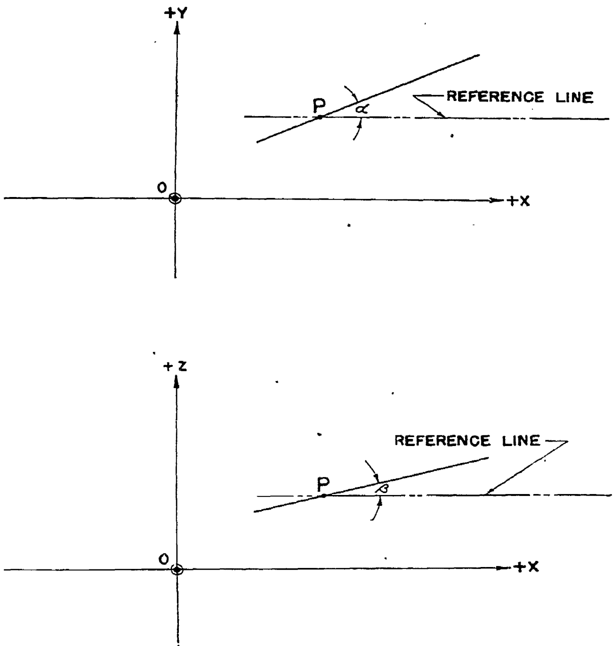


Fig. 4.3.

If the direction ratios of a certain line are $a:1:c$, then a is the tangent of the angle between the y axis and the projection of the line on the xy plane, and c is the tangent of the angle between the y axis and the projection of the line on the yz plane. If the direction ratios are $a:b:1$, then a is the tangent of the angle between the z axis and the projection of the line on the xz plane, and b is the tangent of the angle between the z axis and the projection of the line on the yz plane.

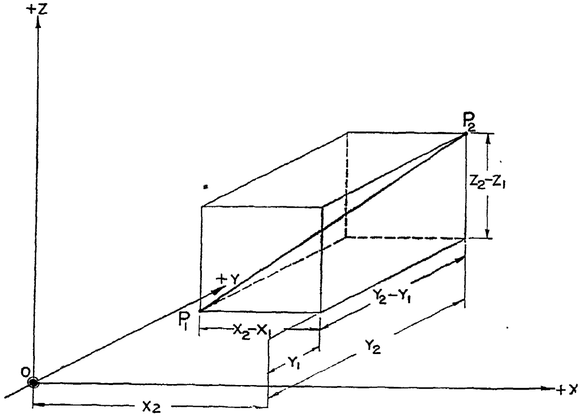


FIG. 4.4.

Corresponding to each line there are an infinite number of direction ratios. For example, 2:3:5 and 4:6:10 can be the direction ratios of the same line. If two lines are parallel their direction ratios are equal or proportional, and conversely.

4.2. True length of a line segment. The true length of a line segment is the diagonal of a "box" of which the differences of the corresponding coordinates are the sides. In this diagram, suppose that P_1 has coordinates (x_1, y_1, z_1) and P_2 has coordinates (x_2, y_2, z_2). Then $x_2 - x_1 = a$, $y_2 - y_1 = b$, and $z_2 - z_1 = c$. Now the length of the diagonal is $\sqrt{a^2 + b^2 + c^2}$, and so the true length of the line segment P_1P_2 is given by the formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 1. Find the true length of the line segment connecting the two points (8, 15, 25) and (5, 11, 13).

$$L = \sqrt{(8 - 5)^2 + (15 - 11)^2 + (25 - 13)^2} = 13.$$

Example 2. Find the true length of the front spar top lofted line as determined by the two points (0, −48.091, 10.738) and (152, −20.040, 6.435).

$$L = \sqrt{(152 - 0)^2 + (-20.040 + 48.091)^2 + (6.435 - 10.738)^2}.$$
$$L = \sqrt{23104 + 786.858601 + 18.515809}.$$
$$L = \sqrt{23909.374410}.$$
$$L = 154.627.$$

Example 3. Find the true length of the engine mount tube AB as shown in Fig. 4.5. Notice that an auxiliary system of axes can be set up to find a true length, which is an isolated problem and is purely local in character.

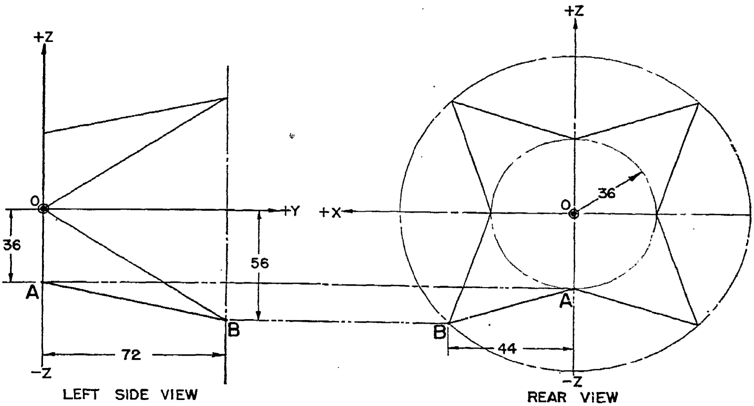


FIG. 4.5.—Engine mount.

First determine the coordinates of the points A and B. They are A(0, 0, −36) and B(44, 72, −56). Then substitute in the formula for the true length of a line segment:

$$L = \sqrt{(44 - 0)^2 + (72 - 0)^2 + (-56 + 36)^2}.$$
$$L = \sqrt{1936 + 5184 + 400}.$$
$$L = \sqrt{7520} = 86.718.$$

Example 4. From the information on the following drawing find the true length of the trailing edge of this twisted wing (see Fig. 4.6).

The coordinates of A can be read directly from the drawing. They are A(0, 98.75, −0.183). The x coordinate of B is 312, its z coordinate is 2.821, but its y coordinate must be calculated. The y coordinate of B will be equal to the difference between 312 tan 26′38″ and 98.75. It is therefore equal to 96.335, *i.e.*, the coordinates of B are B(312, 96.335, 2.821). Substituting in the formula for the length of a line segment,

$$L = \sqrt{(0 - 312)^2 + (98.75 - 96.335)^2 + (-0.183 - 2.821)^2}.$$
$$L = 312.024.$$

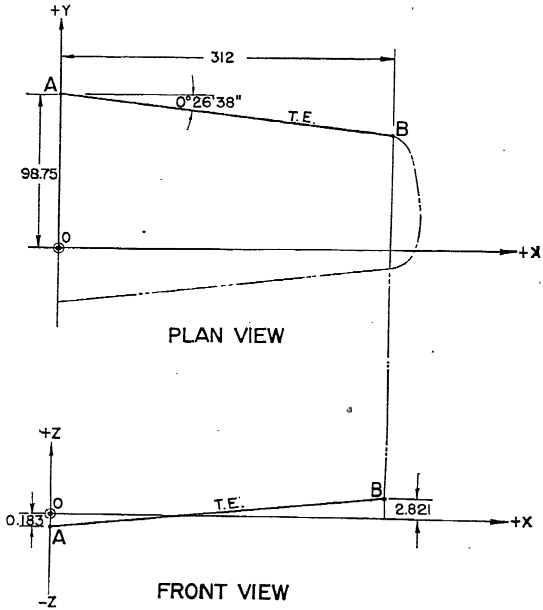

+Y
312
A
0°26'38"
T.E.
B
98.75
0
+X
PLAN VIEW
+Z
B
0
T.E.
2.821
0.183
A
+X
-Z
FRONT VIEW

FIG. 4.6.

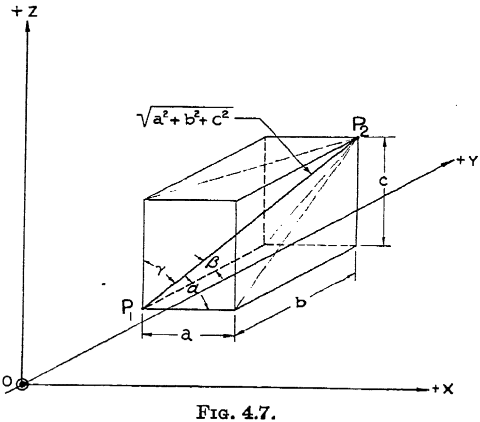

+Z
$\sqrt{a^2+b^2+c^2}$
P_2
+Y
c
β
γ
α
P_1
b
a
O
+X

FIG. 4.7.

This formula is perfectly general and can be used to calculate the true distance between any two points on the airplane, when the coordinates of the two points have been determined.

4.3. Direction cosines of a line. The angle between two directed lines in space, AB and CD, which do not intersect, is defined to be the angle between AB and a line parallel to CD which does intersect AB, or the angle between CD and a line parallel to AB which does intersect CD. The direction cosines of a straight line in space are the cosines of the true angles that the straight line makes with the x axis, y axis, and z axis, respectively. These angles are denoted by α, β, γ, respectively; *i.e.*, α is the true angle between the line and the x axis, β is the true angle between the line and the y axis, and γ is the true angle between the line and the z axis. In Fig. 4.7

$$\cos\alpha = \frac{}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}}.$$

$$\cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Now $a = x_2 - x_1$, $b = y_2 - y_1$, $c = z_2 - z_1$; so,

$$\cos\alpha = \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$\cos\beta = \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$\cos\gamma = \frac{z_2 - z_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

where $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ is the true length of the line segment P_1P_2.

Example 1. Find the direction cosines of the straight line determined by the two points (8, 6, 4) and (2, 3, 6).

First find the direction ratios. They are $6:3:-2$. Then find the length of the line segment. It is exactly 7. Therefore the direction cosines are

$$\cos\alpha = \frac{-}{7}, \qquad \cos\beta \qquad \cos\gamma = \frac{-2}{}$$

Example 2. Find the direction cosines of the flap hinge center line which is determined by two points: its intersection with the normal rib plane at station 60 (60, −40.051, −5.621) and its intersection with the normal rib plane at station 306 (306, −20.036, −4.376).

Find the direction ratios by taking the differences of the x, y, z coordinates in turn. They are 246:20.015:1.245. Then find the true length of the line segment

$$L = \sqrt{(246)^2 + (20.015)^2 + (1.245)^2}.$$
$$L = 246.816.$$

Therefore the direction cosines are

$$\frac{246}{246.816} \quad 0.99669.$$
$$\cos \beta = \frac{20.015}{246.816} \quad 0.08109.$$
$$\cos \gamma = \frac{1.245}{246.816} \quad 0.00504.$$

Example 3. Find the direction cosines of the trailing edge of the twisted wing, which is determined by the two points $A(0,\ 98.75,\ -0.183)$, $B(312,\ 96.335,\ 2.821)$.

First find the direction ratios of the trailing edge. They are

$$312: -2.415: 3.004.$$

Then find the true length as before. It is 312.024. Divide each of the direction ratios by this length. Therefore the direction cosines are

$$\cos \alpha = 0.99992, \qquad \cos \beta = -0.00774, \qquad \cos \gamma = 0.00963.$$

Example 4. The direction ratios of a line are 1:2:2. Find the direction cosines of the line.

$$\cos \alpha = \frac{}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{}{3}.$$
$$\cos \beta = \frac{2}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{2}{3}.$$
$$\cos \gamma = \frac{}{\sqrt{1^2 + 2^2 + 2^2}} \quad \frac{}{3}.$$

If a line is determined by two points the direction cosines can be calculated as in Example 1. If the direction ratios of a line are given (instead of two points on the line) the direction cosines can be calculated as in Example 4. If the direction ratios of a line obtained by subtracting the x coordinates, y coordinates, and

z coordinates in turn are $a{:}b{:}c$, then the direction cosines of the line are

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}$$
$$\cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}.$$
$$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

If a set of direction ratios proportional to $a{:}b{:}c$ is used, such as $Ka{:}Kb{:}Kc$, then

$$\cos\alpha = \frac{Ka}{\sqrt{K^2a^2+K^2b^2+K^2c^2}} = \frac{a}{\sqrt{a^2+b^2+c^2}}$$
$$\cos\beta = \frac{Kb}{\sqrt{K^2a^2+K^2b^2+K^2c^2}} = \frac{b}{\sqrt{a^2+b^2+c^2}}$$
$$\cos\gamma = \frac{Kc}{\sqrt{K^2a^2+K^2b^2+K^2c^2}} = \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

That is, the same answers for the direction cosines are obtained by using any set of direction ratios proportional to the given set of direction ratios.

It can be proved that the sum of the squares of the direction cosines is equal to one. Referring to the formulas for the direction cosines in terms of a, b, c,

$$\cos^2\alpha = \frac{a^2}{a^2+b^2+c^2}$$
$$\cos^2\beta = \frac{b^2}{a^2+b^2+c^2}$$
$$\cos^2\gamma = \frac{c^2}{a^2+b^2+c^2}$$

Therefore $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This is a very useful check on direction cosines. After calculating a set of direction cosines it is always a good idea to check them by squaring and adding. The result should be *one*.

Example 5. If $\cos\alpha = \frac{6}{7}$, $\cos\beta = \frac{3}{7}$, and $\cos\gamma = \frac{-2}{7}$, then the squares are $\frac{36}{49}$, $\frac{9}{49}$, $\frac{4}{49}$. The sum of the squares is one.

Example 6. Determine whether 0.6, 0.5, 0.25 are direction cosines. The squares are 0.36, 0.25, 0.0625. The sum of these squares is not equal to one, and so the original set of numbers could not be direction cosines.

Example 7. The direction cosines of the flap hinge center line were calculated to be 0.99669, 0.08109, 0.00504. Squaring and adding gives one, and the numbers therefore constitute a set of direction cosines.

The direction cosines of a line are the cosines of the true angles between the line and the x axis, y axis, and z axis, respectively. Now the x axis is normal (perpendicular) to the yz plane. The angle between a line and a plane is the complement of the angle between the line and a normal to the plane. Therefore the first direction cosine is the cosine of the complement of the true angle between the line and the yz plane. Now $\cos (90° - A) = \sin A$; *i.e.*, it is the sine of the true angle between the line and the yz plane. Likewise, the second direction cosine is the sine of the true angle between the line and the xz plane. Also, the third direction cosine is the sine of the true angle between the line and the xy plane.

In view of the above remarks, consider a line in the fuselage reference system. The yz plane is parallel to the buttock line planes. The xz plane is parallel to the fuselage station planes. The xy plane is parallel to the water line planes. Therefore the direction cosines of the line are the sines of the true angles between the line and a buttock line plane, a fuselage station plane, and a water line plane, respectively.

Example 8. A certain control cable is determined by the points (24, −56, −8) and (42, 75, 12). Find the true angle between this control cable and water line 6.

Now the water line plane is parallel to the xy plane. The xy plane is normal to the z axis. Therefore the third direction cosine is the one required. The difference of the z coordinates of the two points on the control cable is 20. The distance between the two points is 133.735. The third direction cosine is therefore $\frac{20}{133.735} = 0.14955$. Find this number in a table of sines. The angle is 8°36′2″, which is the required angle.

Example 9. Find the true angle between the control cable in the previous example and fuselage station plane 83.

The fuselage station plane is parallel to the xz plane. The y axis is normal to the xz plane. Therefore the second direction cosine is required. The difference of the y coordinates of the two points that determine the control cable is 131. The distance between these two points is, as before, 133.735.

The second direction cosine is therefore $\frac{131}{133.735}$ 0.97955. Find this number in a table of sines. We find the required angle to be 78°23′30″.

Example 10. A certain wing is lofted by the wing chord plane system. The aileron hinge center line is determined by two points (106.4, 14.036, −6.946) and (240.125, 29.435, −2.427). Find the true angle between this aileron hinge center line and the wing chord plane.

The wing chord plane is the $x_w y_w$ plane. The z_w axis is normal to this plane. Therefore the third direction cosine is the one sought. The difference between the z_w coordinates of the two points is 4.519. The true distance between the two points is 134.685. The third direction cosine is $\frac{4.519}{134.685} = 0.03355$. Finding this number in a table of sines gives the true angle as 1°55′21″.

Since the direction cosines of a line give the true angles between the line and the three reference axes, and since they can be used to find the true angles between the line and the three reference planes, they have a great intrinsic value. However, their main use is to find true angles and distances between points, lines, and planes by methods that will be developed later.

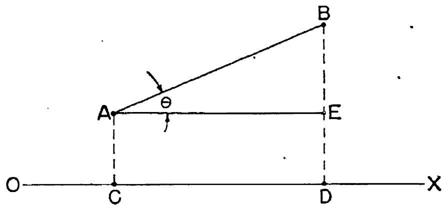


FIG. 4.8.

4.4. Projection Theorems. Consider the line segment AB and the line OX, both of which we assume lie in the same plane (see Fig. 4.8). Drop perpendiculars from A and B to OX, meeting OX at C and D, respectively. The projection of AB on OX is defined to be CD. Draw AE through A parallel to OX. Then $CD = AE$. If θ is the angle between AB and OX, then θ is also the angle between AB and AE. Now $\frac{AE}{AB} = \cos\theta$, so $AE = AB\cos\theta$. But $CD = AE$, so $CD = AB\cos\theta$; *i.e.*, the projection of AB on OX is equal to the length of AB times the cosine of the angle between AB and OX.

Consider the broken line $ABCD$ and the line OX (see Fig. 4.9). Let $ABCD$ and OX lie in the same plane.

As in the preceding paragraph, the projection of AB on OX is EF, the projection of BC on OX is FG, and the projection of CD on OX is GH. Therefore the projection of $ABCD$ on OX is $EF + FG + GH$. But $EF + FG + GH = EH$, and so the projection of $ABCD$ on OX is EH. Now consider the line

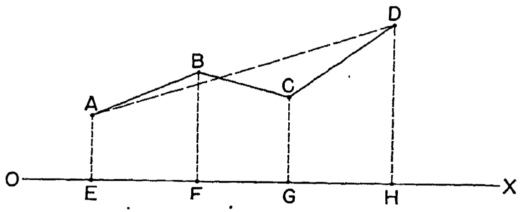


FIG. 4.9.

segment AD. This line segment is called the *closing line segment* for the broken line segment $ABCD$. It joins the first and last points of $ABCD$. The projection of AD on OX is also EH, *i.e.*, the projection of a broken line segment on a given line is equal to the projection of the closing line segment on the given line. This is true for a broken line segment of any finite number of parts. It is also true if some of the segments are as in Fig. 4.10,

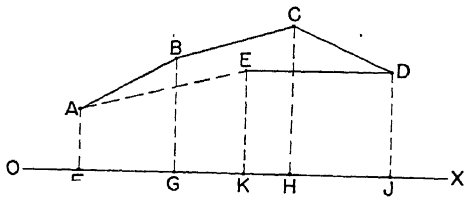


FIG. 4.10.

if the idea of "directed line segments" is used in applying the statements.

In Fig. 4.10 the projection of AB on OX is FG, the projection of BC on OX is GH, the projection of CD on OX is HJ, and the projection of DE on OX is JK. Here $DE = -ED$ and $JK = -KJ$. The projection of $ABCDE$ on OX is equal to $FG + GH + HJ + JK = FG + GH + HJ - KJ = FK$. The

closing line segment of $ABCDE$ is AE. The projection of AE on OX is FK. Therefore the projection of $ABCDE$ on OX is FK.

Similar projection theorems are true for line segments and lines in space. The projection of a point on a plane is the foot of the perpendicular drawn from the point to the plane. The projection of a line segment AB on a given plane is the segment CD joining the projections C and D of the points A and B on the plane. The projection of a point on a given line is the point in which the given line is intersected by a plane that passes through the given point and is perpendicular to the given line. The projection of a line segment AB on a given line OX in space

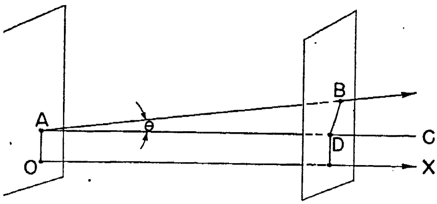


FIG. 4.11.

is the line segment CD joining the projections C and D of A and B on OX.

The length of the projection of a given line segment on a given line is equal to the length of the given line segment multiplied by the cosine of the true angle between the given line segment and the given line (see Fig. 4.11).

In this diagram AB is the given line segment and OX is the given line. AB and OX are directed, as shown by the arrows. Through A draw a line AC parallel to OX. Then the projection of AB on OX is equal to the projection of AB on AC, which is AD. Now $\frac{AD}{AB} = \cos\theta$, and so $AD = AB \cos\theta$; *i.e.*, the length of the projection of AB on OX is equal to the length of AB multiplied by the cosine of the true angle between AB and OX.

If $OABP$ is a broken line segment in space, then the projection of $OABP$ on a given line OX' is equal to the projection of the closing line segment OP on OX' (see Fig. 4.12). Here $OABP$ is a broken line segment and OP is the closing line segment. The projection of OP on OX' is equal to the length of OP times the cosine of the true angle between OP and OX'.

These projection theorems, both in a plane and in space, are extremely helpful in deriving formulas and proving theorems in solid analytic geometry.

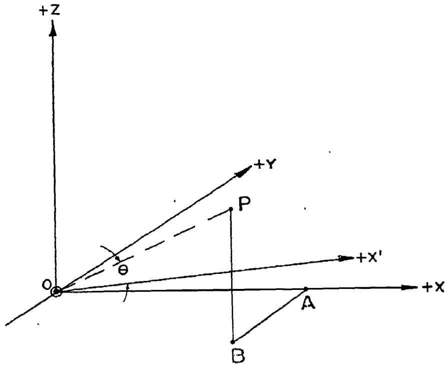


FIG. 4.12.

4.5. Angle between two lines. Consider two directed lines OS and OR which make angles α_1, β_1, γ_1, and α_2, β_2, γ_2 with the

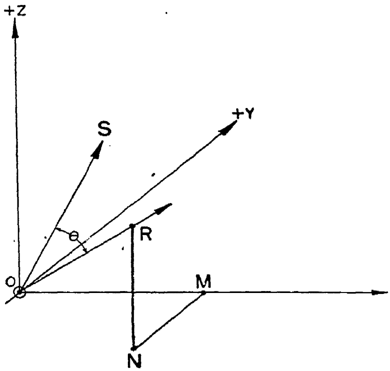


FIG. 4.13.

x, y, z axes, respectively. If the two lines do not intersect, draw parallel to one of them a line that does intersect the other one. For simplicity, and without loss of generality, take the

point of intersection to be the origin of a system of coordinates. Let $OR = r_2$ and let the coordinates of R be

$$x_2 = OM, \qquad = -MN, \qquad z_2 = NR.$$

The projection of the broken line $OMNR$ on OS is equal to the sum of the projections of OM, MN, and NR on OS, and this is in turn equal to the projection of the closing line OR on OS. Therefore

$$OR \cos \theta = OM \cos \alpha_1 + MN \cos \beta_1 + NR \cos \gamma_1.$$

Now

$$OR = r_2, \qquad OM = r_2 \cos \alpha_2, \qquad MN = r_2 \cos \beta_2,$$
$$NR \qquad r_2 \cos \gamma_2.$$

Therefore

$$r_2 \cos \theta = r_2 \cos \alpha_1 \cos \alpha_2 + r_2 \cos \beta_1 \cos \beta_2 + r_2 \cos \gamma_1 \cos \gamma_2.$$
$$\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.$$

That is, if the direction cosines of two lines are a, b, c and d, e, f, respectively, then the angle between the two lines is given by the formula

$$\cos \theta = ad + be + cf.$$

This formula is of extreme importance in problems that involve the angle between two lines. It enables one to calculate mathematically the angle between two lines to a degree of accuracy that is limited only by the accuracy of the basic dimensions that determine the two given lines. It is also the basis of further formulas that will enable one to find mathematically the true angle between a line and a plane and also the true angle between two planes. It is fundamental to much of the work that follows.

Example 1. Calculate the angle between the two lines whose direction cosines are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$ and $\frac{3}{7}$, $\frac{6}{7}$, $\frac{2}{7}$.

$$\cos \theta = (\tfrac{1}{3})(\tfrac{3}{7}) + (\tfrac{2}{3})(\tfrac{6}{7}) + (\tfrac{2}{3})(\tfrac{2}{7}).$$
$$\cos \theta = 0.90476.$$
$$\theta = 25°12'32''.$$

Example 2. Calculate the true angle between the two lines $A(16, 5, 7)$ $B(15, 3, 5)$ and $C(25, 10, 6)$ $D(22, 4, 4)$. First calculate the direction cosines. The direction ratios of AB are $1:2:2$. The length of the segment

AB is 3. The direction cosines of AB are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$. The direction cosines of CD are $\frac{3}{7}$, $\frac{6}{7}$, $\frac{2}{7}$. Therefore

$$\cos\theta = (\tfrac{1}{3})(\tfrac{3}{7}) + (\tfrac{2}{3})(\tfrac{6}{7}) + (\tfrac{2}{3})(\tfrac{2}{7}).$$
$$\cos\theta = 0.90476.$$
$$\theta = 25°12'32''.$$

Example 3. The flap hinge center line is determined by the two points $A(60, -40.051, -5.621)$, $B(306, -20.036, -4.376)$. The aileron hinge center line is determined by the two points $C(106.4, 14.036, -6.946)$, $D(240.125, 29.435, -2.427)$. Calculate the true angle between the flap hinge center line and the aileron hinge center line.

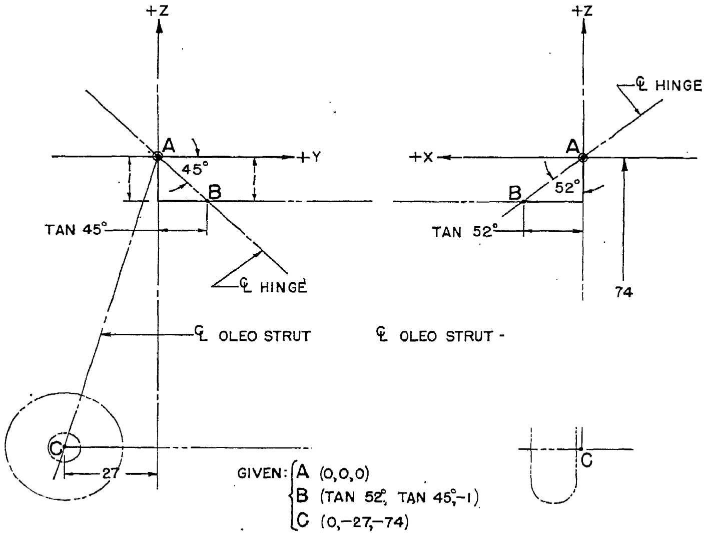


FIG. 4.14.

First calculate the direction cosines of the two center lines. They are, correct to five decimal places,

$$\begin{array}{ll} AB & 0.99669,\ 0.08109,\ 0.00504. \\ CD & 0.99287,\ 0.11433,\ 0.03355. \end{array}$$

Therefore

$$\cos\theta = 0.98958 + 0.00927 + 0.00017.$$
$$\cos\theta = 0.99902.$$
$$\theta = 2°32'0''.$$

Example 4. Calculate the true angle between the center line of oleo strut and the center line of hinge about which the oleo strut rotates, from the data given in Fig. 4.14.

Set up an auxiliary system of axes as shown. The center line of hinge is determined by two points

$$A(0,\ 0,\ 0).$$
$$B(\tan 52°,\ \tan 45°,\ -1).$$

The direction ratios of the center line of hinge are

$$1.2799:1:-1.$$

The length of this segment of the center line of hinge is

$$1.9074.$$

The direction cosines of the center line of hinge are

$$0.67102,\ 0.52427,\ -0.52427.$$

The center line of oleo strut is determined by the two points

$$A(0,\ 0,\ 0).$$
$$C(0,\ -27,\ -74).$$

The direction ratios of the center line of oleo strut are

$$0:27:74.$$

The length of the center line of oleo strut is

$$78.772.$$

The direction cosines of the center line of oleo strut are

$$0:0.34276:0.93942.$$

The true angle between the center line of hinge and the center line of oleo strut is given by

$$\cos\theta = 0 + 0.17970 - 0.49251.$$
$$\cos\theta = -0.31281.$$
$$\theta = 108°13'43''.$$

If the direction ratios of one given line are r, s, t and the direction ratios of another given line are u, v, w, then the true angle between these two lines is given by

$$\cos\theta = \frac{ru + sv + tw}{\sqrt{r^2 + s^2 + t^2}\,\sqrt{u^2 + v^2 + w^2}}.$$

This formula can be derived as follows: The direction cosines of the two given lines are

$$\frac{r}{\sqrt{r^2 + s^2 + t^2}}, \qquad \frac{s}{\sqrt{r^2 + s^2 + t^2}}, \qquad \frac{t}{\sqrt{r^2 + s^2 + t^2}}.$$
$$\frac{u}{\sqrt{u^2 + v^2 + w^2}} \qquad \frac{v}{\sqrt{u^2 + v^2 + w^2}}, \qquad \frac{w}{\sqrt{u^2 + v^2 + w^2}}.$$

The cosine of the true angle between the two given lines is equal to the sum of the products of the corresponding direction cosines. Multiplying and adding these direction cosines, and collecting terms over the common denominator, we have the above formula, expressed in terms of direction ratios.

Example 5. The direction ratios of one line are 1:2:2 and the direction ratios of another line are 3:6:2. Find the true angle between the two lines

$$\cos\theta = \frac{(1)(3) + (2)(6) + (2)(2)}{\sqrt{1^2 + 2^2 + 2^2}\sqrt{3^2 + 6^2 + 2^2}}.$$

$$\cos\theta = 0.90476.$$

$$\theta = 25°12'32''.$$

Compare this example with Example 1.

This method of finding the true angle between two lines when the direction ratios are given is very convenient and will be used often in the succeeding chapters.

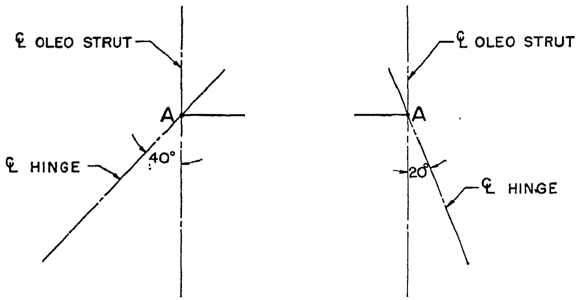


FIG. 4.15.

Sometimes the information as given on the engineering drawing is such that the line is not directly determined by two points. An example of such a case is when the line is determined by one point and the projected angles in two views (see Fig. 4.15). In this case a good procedure is as follows (see Fig. 4.16): Set up an auxiliary system of axes as shown. Assume a unit distance as shown on the z axis. Now $BC = \tan 40°$, and $BD = \tan 20°$. Therefore the coordinates of the point B on the given line AB are $(-\tan 20°, -\tan 40°, -1)$, *i.e.*, the coordinates of B are

$(-0.36397, \ -0.83910, \ -1)$. Also, the coordinates of A are $(0, 0, 0)$. The direction ratios of AB are

$$0.36397:0.83910:1.$$

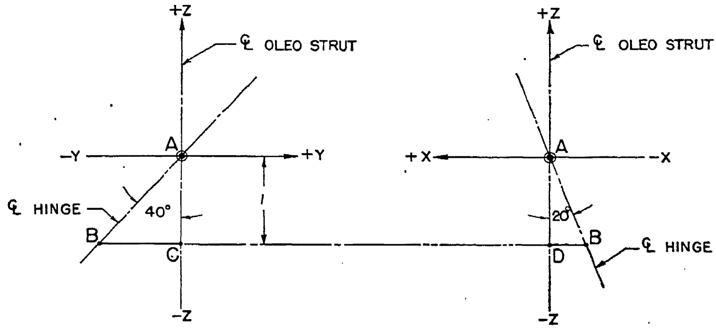


Fig. 4.16.

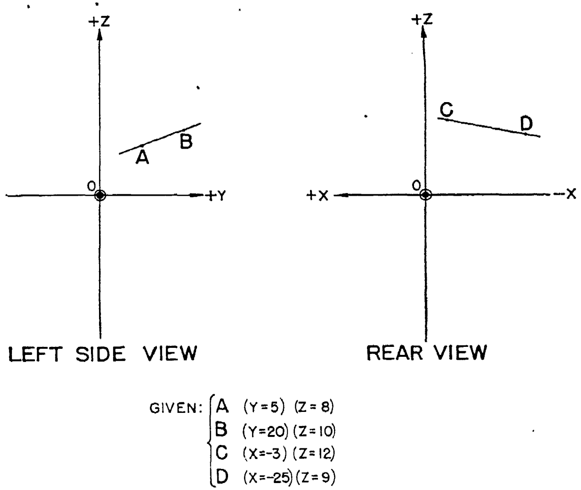


Fig. 4.17.

The length of the segment AB is 1.3552. Therefore the direction cosines of AB are 0.26857, 0.61917, 0.73790.

The idea of assuming a unit distance and using the trigonometric functions to determine the coordinates of a point on the line,

when the projected angles are given, is important and should be mastered and used when possible.

Sometimes two points on a line are given in one of the basic views and two other points on the same line are given in another related basic view (see Fig. 4.17). This diagram shows two views of the same line. Suppose that in the left-side view the coordinates of A are $y = 5$ and $z = 8$, and the coordinates of B are $y = 20$ and $z = 10$. Suppose that in the rear view the coordinates of C are $x = -3$ and $z = 12$, and the coordinates of D are $x = -25$ and $z = 9$. First write the equation of the line CD in the rear view. The equation is

$$3x - 22z + 273 = 0.$$

To find the coordinates of A and B in the rear view, substitute $z = 8$ and $z = 10$ in the equation of the line. This gives $x = -32\frac{1}{3}$ and $x = -17\frac{2}{3}$, respectively. Therefore the coordinates of A are $(-32\frac{1}{3}, 5, 8)$ and the coordinates of B are $(-17\frac{2}{3}, 20, 10)$. Having thus determined the two points that determine the line, proceed as in the usual way to find the direction cosines of AB.

Another method is to calculate the slopes of AB and CD with respect to the positive z axis. Then the direction ratios are $\tan \alpha : \tan \beta : 1$, where α and β are the inclinations of CD and AB with respect to the z axis.

4.6. Directed lines and true angles. A line connecting two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ can be assigned a direction as follows: The direction of the line as indicated by P_1P_2 is from P_1 to P_2 and its direction ratios are $(x_2 - x_1):(y_2 - y_1):(z_2 - z_1)$. Similarly the direction of line P_2P_1 is from P_2 to P_1, and its direction ratios are $(x_1 - x_2):(y_1 - y_2):(z_1 - z_2)$.

Example 1. Find the direction ratios of the line as determined by two points $A(16, -4, 13)$ and $B(4, -1, 9)$.

The direction ratios of line $AB = -12:3:-4$.

The direction ratios of line $BA = 12:-3:4$.

Notice from this example that, as a point moves along the line AB from A to B its change in $x = -12$, its change in $y = 3$, and its change in $z = -4$; and conversely that as a point moves along the line BA from B to A its change in $x = 12$, its change in $y = -3$, and its change in $z = 4$.

The direction cosines of line AB are

$$\cos\alpha = -\tfrac{12}{13}, \qquad \cos\beta = \tfrac{3}{13}, \qquad \cos\gamma = -\tfrac{4}{13}.$$

The direction cosines of line BA are

$$\cos\alpha = \tfrac{12}{13}, \qquad \cos\beta = -\tfrac{3}{13}, \qquad \cos\gamma = \tfrac{4}{13}.$$

Thus it can be seen that the direction cosines of a given line are the cosines of the angles that the given line makes with the positive directions of the x, y, and z axes, respectively, and that it is possible to have two solutions to the problem of finding the

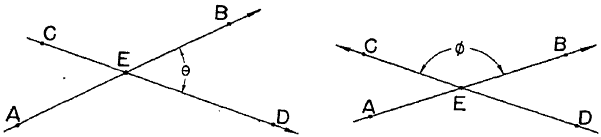


FIG. 4.18.

angle between two lines unless the directions are specified (see Fig. 4.18). Notice, however, that the two angles are supplementary and that their cosines therefore are opposite in sign.

Example 2. Find the angle between the lines AB (as defined in Example 1) and the line CD determined by two points $C(10, -4, 5)$ and $D(4, -2, 8)$.

The direction ratios of line AB are $-12:3:-4$.
The direction ratios of line CD are $-6:2:3$.
The direction cosines of line AB are $-\frac{12}{13}, \frac{3}{13}, -\frac{4}{13}$.
The direction cosines of line CD are $-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$.

$$\begin{aligned} \cos\theta &= \tfrac{72}{91} + \tfrac{6}{91} - \tfrac{12}{91} = \tfrac{66}{91} = 0.72527. \\ \theta &= 43°30'30''. \end{aligned}$$

Example 3. Find the angle between the lines BA and DC (Fig. 4.18 and Example 2).

The direction cosines of line BA are $\frac{12}{13}, -\frac{3}{13}, \frac{4}{13}$.
The direction cosines of line DC are $\frac{6}{7}, -\frac{2}{7}, -\frac{3}{7}$.

$$\begin{aligned} \cos\angle AEC &= \tfrac{72}{91} + \tfrac{6}{91} - \tfrac{12}{91} = \qquad = 0.72527. \\ \angle AEC &= 43°30'30''. \end{aligned}$$

Notice that the angles of Examples 2 and 3 are equal. This is true because the angles are vertical angles (see Fig. 4.18).

Let us check by using this method as outlined, if the angle as determined by lines AB and DC is the supplement of angle θ as shown in Fig. 4.18.

The direction cosines of line AB are $-\frac{12}{13}, \frac{3}{13}, -\frac{4}{13}$.

The direction cosines of line DC are $\frac{6}{7}, -\frac{2}{7}, -\frac{3}{7}$.

$$\cos \phi = -\tfrac{72}{91} - \tfrac{6}{91} + \tfrac{12}{91} = -\tfrac{66}{91} = -0.72527.$$
$$\phi = 136°29'30''.$$

Angle ϕ is thus proved to be the supplement of angle θ. It can be seen now that, in order to solve for the angle between

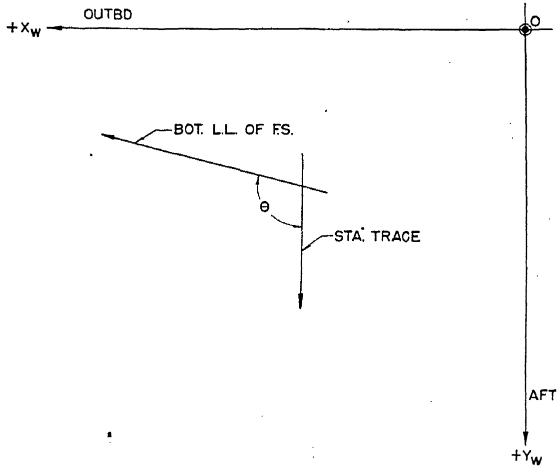


FIG. 4.19.

two lines, it is necessary to specify which angle, since there are two solutions, one being the supplement of the other.

Example 4. Find the true angle in the lower plane of the front spar cap between the bottom lofted line of the front spar, and a normal wing rib station trace. State whether the angle in the aft outboard corner is an open or closed angle.

Given: Direction ratios of bottom lofted line of front spar:

$$(x_w):(y_w):(z_w) = (1):(-0.08916):(0.01472).$$

Direction ratios of normal wing rib station trace in the lower front spar cap plane:

$$(x_w):(y_w):(z_w) = (0):(1):(-0.11277).$$

Note: From given direction ratios we are solving for the aft outboard angle (see Fig. 4.19).

$$\cos = \frac{-0.08916 + (0.01472)(-0.11277)}{\sqrt{1 + (-0.08916)^2 + (0.01472)^2}\sqrt{0 + 1 + (-0.11277)^2}}$$

$$= \frac{-0.08916 - 0.00166}{(1.00407)(1.00634)} = \frac{-0.09082}{1.01044} = -0.08988.$$

$$\theta = 95°9'25''.$$

(Angle is open 5°9'25″ in aft outboard corner.)

CHAPTER 5

TRUE LENGTHS AND TRUE ANGLES (*Continued*)

Direction ratios and direction cosines are so useful that it is advisable to calculate and tabulate for future reference the direction ratios and direction cosines of all the basic straight lines on the airplane. This procedure saves the time and trouble of recalculating these values every time a situation arises in which they are needed. The term *basic lines* as used here includes the various systems of axes, hinge lines, lofted lines (mold lines), etc.

The problem of determining the true angle between two lines, as described in Chap. 4, is but one of many similar problems that can be solved very simply and accurately by solid analytic geometry. These other problems include the true angle between a line and a plane, the true angle between two planes, the shortest distance between two lines, and many more problems. Some of these applications will be discussed in this chapter.

The importance of establishing and controlling certain basic dimensions cannot be overemphasized. From the basic dimensions other dimensions are calculated. The accuracy of the basic dimensions limits the accuracy of the calculated dimensions. The same is true for layout techniques. The accuracy of the completed layout depends upon the accuracy of the information upon which the layout is based. The control of basic dimensions is especially critical in mass production of airplanes and in cases where subassemblies are subcontracted to other companies or other plants of the same company. The matter of controlling basic dimensions is complex and involves materials, properties of materials, manufacturing processes, heat-treatment of metals, tools, tool design, jig building, and many other items. Its importance is so great that carefully planned steps must be taken to ensure that it receives the consideration that it warrants.

5.1. Direction ratios and direction cosines of axes with respect to themselves. The direction cosines of a line are the cosines of the true angles between the line and the three mutually perpendicular axes. Consider the x, y, z system of axes in rigged

position. The x axis makes an angle of 0° with itself, the x axis makes an angle of 90° with the y axis, and the x axis makes an angle of 90° with the z axis. The true angles between the x axis and the x axis, y axis, and z axis are therefore 0°, 90°, and 90°, respectively. The cosines of these angles are 1, 0, 0. Therefore the direction cosines of the x axis in rigged position are 1, 0, 0. The numbers can also serve as direction ratios, and the direction ratios of the x axis in rigged position are therefore 1:0:0. The y axis makes true angles 90°, 0°, 90° with the x axis, y axis, and z axis, respectively. The cosines of these angles are 0, 1, 0. Therefore the direction cosines of the y axis in rigged position are 0, 1, 0 and its direction ratios are 0:1:0. The z axis makes true angles 90°, 90°, 0° with the x axis, y axis, and z axis, respectively. The cosines of these angles are 0, 0, 1. The direction cosines of the z axis are therefore 0, 0, 1 and its direction ratios are 0:0:1. These results can be tabulated in the form of a box.

	x	y	z
x	1	0	0
y	0	1	0
z	0	0	1

Consider the wing system of axes, denoted by x_w, y_w, z_w. In both the chord plane type and wing reference plane type the direction cosines of the x_w, y_w, z_w axes *with respect to themselves* are the same as for the rigged axes with respect to the rigged axes, namely, 1, 0, 0; 0, 1, 0; 0, 0, 1. The direction ratios are 1:0:0; 0:1:0; 0:0:1. This is true of any set of axes *with respect to themselves.* Another way of saying the same thing is that the direction cosines of the rigged axes in rigged position, the direction cosines of the wing axes in wing position, the direction cosines of the nacelle axes in nacelle position, etc., are 1, 0, 0; 0, 1, 0; 0, 0, 1.

5.2. Direction ratios and direction cosines of wing reference plane axes with respect to rigged axes. Consider the problem of determining the direction cosines of a set of wing reference plane axes in rigged position, *i.e.*, with respect to the rigged system of axes (see Figs. 5.1 and 3.12).

The x_w axis makes true angles ϕ, 90°, 90° $- \phi$ with the x axis, y axis, and z axis, respectively. The cosines of these angles are $\cos \phi$, 0, $\sin \phi$. Therefore the direction cosines of the x_w axis with respect to the rigged system of axes are $\cos \phi$, 0, $\sin \phi$, and its direction ratios are $\cos \phi : 0 : \sin \phi$. The direction ratios can be reduced by dividing by $\cos \phi$ to give $1:0:\tan \phi$. The y_w axis makes true angles 90°, 0°, 90° with the x axis, y axis, and z axis,

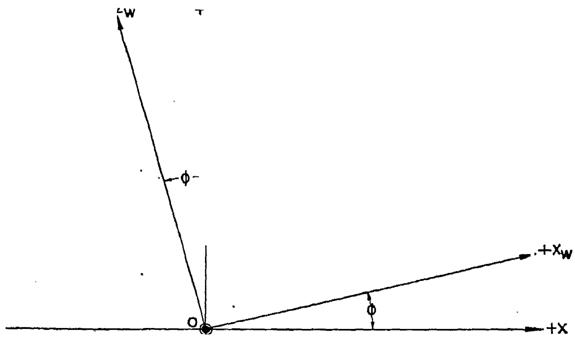


FIG. 5.1.

respectively. The cosines of these angles are 0, 1, 0. Therefore the direction cosines of the y_w axis with respect to the rigged system of axes are 0, 1, 0, and its direction ratios are $0:1:0$. The z_w axis makes true angles 90° + ϕ, 90°, ϕ with the x axis, y axis, and z axis, respectively. The cosines of these angles are $- \sin \phi$, 0, $\cos \phi$. Therefore the direction cosines of the z_w axis with respect to the rigged system of axes are $- \sin \phi$, 0, $\cos \phi$ and its direction ratios are $- \sin \phi : 0 : \cos \phi$. The direction ratios can be reduced by dividing by $\cos \phi$ to give $- \tan \phi : 0 : 1$. These direction cosines may be tabulated as follows:

	x	y	z
x_w	$\cos \phi$	0	$\sin \phi$
y_w	0	1	0
z_w	$- \sin \phi$	0	$\cos \phi$

Reading this table by horizontal columns, the direction cosines of the x_w axis are cos ϕ, 0, sin ϕ; the direction cosines of the y_w axis are 0, 1, 0; and the direction cosines of the z_w axis are $-$ sin ϕ, 0, cos ϕ. These are all with respect to the rigged axes x, y, z.

Reading the table by vertical rows, the direction cosines of the x axis are cos ϕ, 0, $-$ sin ϕ; the direction cosines of the y axis are 0, 1, 0; and the direction cosines of the z axis are sin ϕ, 0, cos ϕ. These are all with respect to the wing reference plane axes.

The necessity for knowing the direction cosines of the wing reference plane axes in rigged position is illustrated in the following example.

Example. Find the true angle between a control cable whose direction cosines with reference to the rigged axes are 0.85714, 0.28571, 0.42857 and a normal to the wing reference plane, the angle of dihedral ϕ being 3°.

The wing reference plane is the $x_w y_w$ plane. The z_w axis is normal to the wing reference plane. The direction cosines of the z_w axis, as obtained from the "box," are $-$ sin ϕ, 0, cos ϕ. These are with respect to the rigged axes. Since the angle ϕ is 3°, the direction cosines of the z_w axis are -0.05234, 0, 0.99863. Since the direction cosines of the control cable and the direction cosines of the z_w axis are now both with respect to the rigged axes, the true angle between these two lines can be calculated as follows:

$$\cos \alpha = (0.85714)(-0.05234) + (0.28571)(0) + (0.42857)(0.99863).$$
$$\cos \alpha = 0.38312.$$
$$\alpha = 67°28'22''.$$

5.3. Direction ratios and direction cosines of wing chord plane axes with respect to rigged axes. Consider the problem of determining the direction ratios and direction cosines of a set of wing chord plane axes with respect to a set of rigged axes (see Figs. 5.2 and 3.9). Consider the point P one unit from the origin on the positive x_w axis, the point Q one unit from the origin on the y_w axis, and the point R one unit from the origin on the z_w axis. The true angle between the x_w axis and the x axis is ϕ, the angle of dihedral, and the true angle between the y_w axis and the y axis is θ, the angle of incidence. Notice that the angle of incidence, θ, is measured in the yz plane, which is the plane of symmetry.

The coordinates of P with respect to the x axis, y axis, and z axis are OA, BP, and AB, respectively. In the triangle OAP, OP is one unit, and so OA = cos ϕ. In the triangle OAP, OP is one unit, and so AP = sin ϕ. Now in the triangle ABP, the

angle at B is a right angle, and the sides of angle PAB are perpendicular to the sides of angle θ in triangle ODQ, and so angle $PAB = \theta$. Also, in the triangle ABP, $\dfrac{BP}{AP} = \sin\ \theta$. Since $AP = \sin\ \phi$, then $\dfrac{BP}{\sin\ \phi} = \sin\ \theta$, and $BP = \sin\ \phi\ \sin\ \theta$. Again, in the triangle ABP, $\dfrac{AB}{AP} = \cos\ \theta$. Since $AP = \sin\ \phi$, then

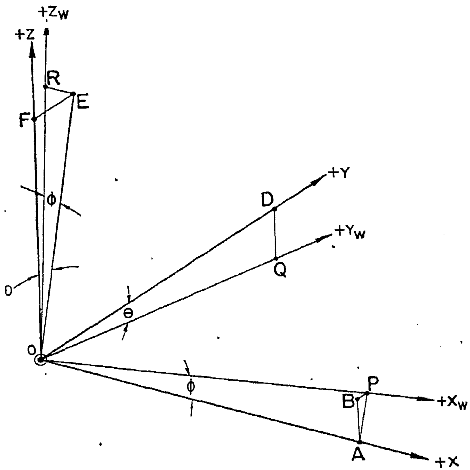


FIG. 5.2.

$\dfrac{AB}{\sin\ \phi} = \cos\ \theta$, and $AB = \sin\ \phi\ \cos\ \theta$. Since the coordinates of P are OA, BP, and AB, therefore the coordinates of P are $\cos\ \phi$, $\sin\ \phi\ \sin\ \theta$, and $\sin\ \phi\ \cos\ \theta$.

The coordinates of O are (0, 0, 0) and the coordinates of P are $(\cos\ \phi, \sin\ \phi\ \sin\ \theta, \sin\ \phi\ \cos\ \theta)$, and the direction ratios of OP are therefore the differences of these values, namely,

$$\cos\ \phi : \sin\ \phi\ \sin\ \theta : \sin\ \phi\ \cos\ \theta.$$

Now $(\cos\ \phi)^2 + (\sin\ \phi\ \sin\ \theta)^2 + (\sin\ \phi\ \cos\ \theta)^2 = \cos^2\ \phi + \sin^2\ \phi\ \sin^2\ \theta + \sin^2\ \phi\ \cos^2\ \theta = \cos^2\ \phi + \sin^2\ \phi\ (\sin^2\ \theta + \cos^2\ \theta) = \cos^2\ \phi + \sin^2\ \phi\ (1) = \cos^2\ \phi + \sin^2\ \phi = 1$. Therefore $\cos\ \phi$, $\sin\ \phi\ \sin\ \theta$, $\sin\ \phi\ \cos\ \theta$ are the direction cosines of OP, since the sum of their

squares is equal to 1 ($OP = 1$, since P was taken as a unit distance along the x_w axis).

The coordinates of Q with respect to the x axis, y axis, and z axis are 0, OD, and DQ, respectively. In the triangle ODQ, OQ is one unit long and the angle ODQ is a right angle, and so $OD = \cos\theta$. Also, $DQ = \sin\theta$. Therefore the coordinates of Q are $0, \cos\theta, -\sin\theta$. Since the coordinates of O are $(0, 0, 0)$ and the coordinates of Q are $(0, \cos\theta, -\sin\theta)$, the direction ratios of OQ are the differences of these values, namely,

$$0 : \cos\theta : -\sin\theta.$$

Now $(0)^2 + (\cos\theta)^2 + (-\sin\theta)^2 = \cos^2\theta + \sin^2\theta = 1$, and so $0, \cos\theta, -\sin\theta$ are the direction cosines of OQ, because the sum of their squares is equal to one ($OQ = 1$, since Q was taken as a unit distance along the y_w axis).

The coordinates of R with respect to the x axis, y axis, and z axis are ER, FE, and OF, respectively. In the triangle OER, OR is one unit, and so $ER = \sin\phi$. In the triangle OFE, the angle at F is a right angle, the sides of angle EOF are perpendicular to the sides of angle θ in triangle ODQ, and so the angle $EOF = \theta$. Also, in the triangle EOF, $\dfrac{FE}{OE} = \sin\theta$. Since $OE = \cos\phi$, then $\dfrac{FE}{\cos\phi} = \sin\theta$, and $FE = \sin\theta\cos\phi$. Again, in the triangle EOF, $\dfrac{OF}{OE} = \cos\theta$. Since $OE = \cos\phi$, then $\dfrac{OF}{\cos\phi} = \cos\theta$, and $OF = \cos\phi\cos\theta$. Therefore the coordinates of R are $-\sin\phi$, $\sin\theta$, $\cos\phi$, and $\cos\phi\cos\theta$.

The coordinates of O are $(0, 0, 0)$ and the coordinates of R are $(-\sin\phi, \sin\theta\cos\phi, \cos\phi\cos\theta)$, and the direction ratios of OR are therefore the differences of these values, namely,

$$-\sin\phi : \sin\theta\cos\phi : \cos\phi\cos\theta.$$

Now $(-\sin\phi)^2 + (\sin\theta\cos\phi)^2 + (\cos\phi\cos\theta)^2 = \sin^2\phi + \sin^2\theta\cos^2\phi + \cos^2\phi\cos^2\theta = \sin^2\phi + \cos^2\phi(\sin^2\theta + \cos^2\theta) = \sin^2\phi + \cos^2\phi(1) = \sin^2\phi + \cos^2\phi = 1$. Therefore $-\sin\phi$, $\sin\theta\cos\phi$, $\cos\phi\cos\theta$ are the direction cosines of OR, since the sum of their squares is equal to one ($OR = 1$, since R was taken as a unit distance along the z_w axis).

The direction cosines of OP, OQ, and OR are the direction cosines of the x_w axis, the y_w axis, and the z_w axis, respectively, with relation to the x, y, z axes. We can arrange these direction cosines in a box:

	x_w	y_w	z_w
x	$\cos\phi$	0	$-\sin\phi$
y	$\sin\phi\sin\theta$	$\cos\theta$	$\cos\phi\sin\theta$
z	$\sin\phi\cos\theta$	$-\sin\theta$	$\cos\phi\cos\theta$

Reading this table by vertical columns, the direction cosines of the x_w axis are $\cos\phi$, $\sin\phi\sin\theta$, $\sin\phi\cos\theta$; the direction cosines of the y_w axis are 0, $\cos\theta$, $-\sin\theta$; and the direction cosines of the z_w axis are $-\sin\phi$, $\sin\theta\cos\phi$, $\cos\phi\cos\theta$. These are all with respect to the rigged axes.

Reading the table by horizontal rows, the direction cosines of the x axis are $\cos\phi$, 0, $-\sin\phi$; the direction cosines of the y axis are $\sin\phi\sin\theta$, $\cos\theta$, $\sin\theta\cos\phi$; and the direction cosines of the z axis are $\sin\phi\cos\theta$, $-\sin\theta$, $\cos\phi\cos\theta$. These are all with respect to the wing chord plane axes.

It should be noted that the values of these basic direction cosines are dependent upon the relative positions of the axes. For example, if the positive direction on the y axis were taken forward instead of aft, the values would be different. So it can be seen that, when finding the direction cosines of the axes with respect to themselves (Art. 5.1) or when finding the direction cosines of one set of axes with relation to another set of axes (Arts. 5.2 and 5.3), it is always important to use the positive direction of all axes involved.

The wing chord plane axes are obtained from the rigged axes by two angles of rotation, θ and ϕ. In various other subassembly situations the two sets of reference axes are related in a similar way. In such a case the above explanation can be followed as a typical example, and the direction cosines of each set of axes with respect to the other can be calculated. We shall discuss these matters in more detail when we study rotation of axes.

5.4. True angle between a line and a plane. The angle between a line and a plane is equal to the complement of the

angle between the line and a normal to the plane (see Fig. 5.3). The angle between the line and the plane in Fig. 5.3 is α. The angle between the line and a normal to the plane is β. Notice that $\alpha + \beta = 90°$, and so the angle α is the complement of the angle β. In order to find the true angle between a line and a plane we can therefore find the true angle between the line and a normal to the plane, and then subtract the answer from 90°. The true angle between the line and the normal to the plane is a

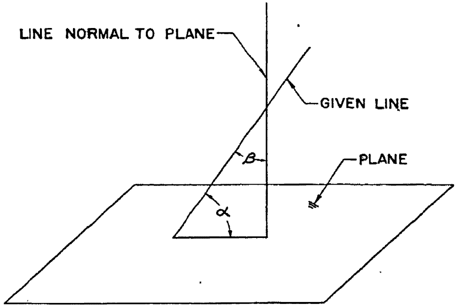


FIG. 5.3.

matter of finding the true angle between two lines, which was explained in Chap. 4.

Example 1. Find the true angle between a line whose direction cosines are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$ and a certain plane, a normal to this plane having direction cosines $\frac{6}{7}$, $\frac{2}{7}$, $\frac{3}{7}$.

The angle between the line and the normal is given by

$$\begin{aligned}\cos\beta &= (\tfrac{1}{3})(\tfrac{6}{7}) + (\tfrac{2}{3})(\tfrac{2}{7}) + (\tfrac{2}{3})(\tfrac{3}{7}).\\ \cos\beta &= \tfrac{16}{21}.\\ \cos\beta &= 0.76190.\\ \beta &= 40°22'6''.\end{aligned}$$

The angle between the line and the plane is therefore

$$90° - 40°22'6'' = 49°37'54''.$$

Since $\cos(90° - A) = \sin A$, an alternative method would be to find 0.76190 in the table of sines, rather than cosines. This would eliminate the operation of subtracting 40°22′6″ from 90° and would give the angle directly as 49°37′54″.

Example 2. Find the true angle between the center line of aileron hinge, whose direction cosines are 0.99316, 0.11433, and 0.02355, and the wing

reference plane. The direction cosines of a normal to the wing reference plane are -0.00759, 0, 0.99997.

The true angle between the center line of aileron hinge and the normal to the wing reference plane is given by

$$\cos \theta = (0.99316)(-0.00759) + (0.11433)(0) + (0.02355)(0.99997).$$
$$\cos \theta = 0.01601.$$
$$\theta = 89°4'58''.$$

Subtract this angle from 90°. The result is 0°55′2″, which is the true angle between the center line of aileron hinge and the wing reference plane.

Example 3. Find the true angle between a control cable determined by the two points (24, -56, -8) and (42, 75, 12) and the plane of a fuselage canted bulkhead as shown in Fig. 5.4.

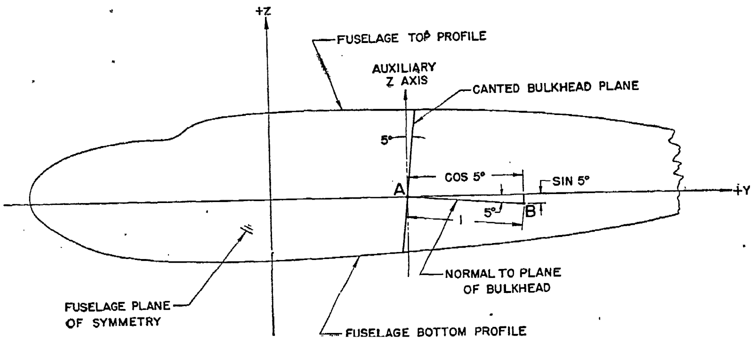


FIG. 5.4.

The direction ratios of the line determining the control cable are 18 : 131 : 20. The distance between the two points is 133.735. The direction cosines of the line determining the control cable are therefore 0.13459, 0.97955, 0.14955. Figure 5.4 illustrates a method for finding the direction cosines of a normal to the canted bulkhead. The coordinates of A are (0, 0, 0). The coordinates of B are (0, cos 5°, $-$ sin 5°) or (0, 0.99619, -0.08716). The direction ratios of AB, obtained in the usual way by subtracting the x coordinates, y coordinates, and z coordinates, are $0 : 0.99619 : -0.08716$. The distance between the points A and B is 1 (see Fig. 5.4). The direction cosines of AB, a normal to the canted bulkhead, are 0, 0.99619, -0.08716. The true angle between the cable and the normal is given by

$$\cos \theta = (0.13459)(0) + (0.97955)(0.99619) + (0.14955)(-0.08716).$$
$$\cos \theta = 0.96278.$$
$$\theta = 15°40'53''.$$

Subtract this angle from 90°. The result is 74°19′7″. This is the true angle between the control cable and the canted bulkhead.

5.5. True angle between two planes. When the direction cosines of normals to two given planes are known, the true angle between the two planes can then be calculated as follows: The angle between two planes is equal to the angle between normals to the two planes (see Fig. 5.5). Thus the problem of finding the true angle between two planes is equivalent to the problem of finding the true angle between two lines.

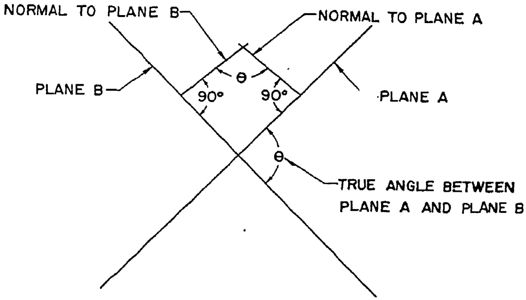


FIG. 5.5.

Example. The direction cosines of a normal to a certain plane are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$. The direction cosines of a normal to another plane are $\frac{3}{7}$, $\frac{6}{7}$, $\frac{2}{7}$. Find the true angle between the two planes.

The true angle between the two normals is given by

$$\cos\theta = (\tfrac{1}{3})(\tfrac{3}{7}) + (\tfrac{2}{3})(\tfrac{6}{7}) + (\tfrac{2}{3})(\tfrac{2}{7}).$$
$$\cos\theta = 0.90476.$$
$$\theta = 25°12'32''.$$

The true angle between the two planes is also 25°12′32″.

When the direction cosines of normals to the two planes are not given directly, they can be calculated by a procedure developed in the next article. Then the procedure is the same as in the foregoing example.

5.6. Calculation of the direction ratios of a normal to a given plane. A line that is perpendicular to two lines in a plane is perpendicular to the plane (see Fig. 5.6).

Suppose that the direction ratios of a line in a certain plane are $a:b:c$, and the direction ratios of another line in the same plane

are $d:e:f$. Assume that the direction ratios of a normal to the plane determined by the two lines are $L:M:N$. Then

$$La + Mb + Nc = 0.$$
$$Ld + Me + Nf = 0.$$

These equations are true because the normal is perpendicular to each of the two lines, two lines are perpendicular when the true angle between them is 90°, and the cosine of 90° is zero.

It can be shown that a set of values for $L:M:N$ are

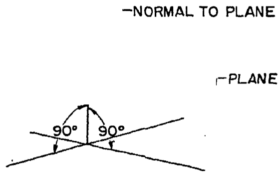


FIG. 5.6.

$$L = bf - ce.$$
$$M = cd - af.$$
$$N = ae - bd.$$

A line normal to two given lines $a:b:c$ and $d:e:f$ can be denoted by $P:Q:1$. This is true since a given set of direction ratios can be divided by one of the numbers in the ratio to give $P:Q:1$. That the line whose direction ratios are $a:b:c$ is perpendicular to the required line $P:Q:1$ can be stated by $aP + bQ + c = 0$. Also it can be stated by $dP + eQ + f = 0$ that the line with given direction ratios $d:e:f$ is perpendicular to the required line whose direction ratios are $P:Q:1$.

Solving simultaneously

$$\begin{aligned} -d &\left\{ aP + bQ = -c, \right. \\ a &\left\{ dP + eQ = -f. \right. \\ Q(ae - bd) &= cd - af. \\ Q &= \frac{cd - af}{ae - bd}. \end{aligned} \qquad \begin{aligned} e &\left\{ aP + bQ = -c, \right. \\ -b &\left\{ dP + eQ = -f. \right. \\ P(ae - bd) &= bf \quad ce. \\ P &= \frac{bf - ce}{ae - bd}. \end{aligned}$$

The required direction ratios are

$$\frac{bf - ce}{ae - bd} : \frac{cd - af}{ae - bd} : 1 \qquad \text{or} \qquad (bf - ce):(cd - af):(ae - bd).$$

To verify this, merely substitute these values in the pair of equations above. The result is

$$(bf - ce)a + (cd - af)b + (ae - bd)c = 0.$$
$$(bf - ce)d + (cd - af)e + (ae - bd)f = 0.$$

These equations reduce to

$$abf - ace + bcd - abf + ace - bcd = 0.$$
$$bdf - cde + cde - aef + aef - bdf = 0.$$

Because these are identities, the values of $L:M:N$ given above are legitimate direction ratios for a normal to the plane.

Notice that the expressions $bf - ce$, $cd - af$, $ae - bd$ can be obtained by writing the six numbers a, b, c, d, e, f as follows:

$$\begin{matrix} a & b & c \\ d & e & f \end{matrix}$$

and then cross-multiplying in the following way: First cover a and d, the first vertical column, and cross-multiply in the order bf–ce. This is similar to the cross multiplication used in evaluating second-order determinants. Next, cover b and e, the second vertical column, and cross-multiply "backward" in the order cd–af. This is the opposite order of the cross multiplication in determinants. Last, cover the numbers c and f, the third vertical column, and cross-multiply in the order ae–bd. This is similar to the cross multiplication used in evaluating second-order determinants.

Example 1. The direction ratios of a certain line are 2:3:4, and the direction ratios of another line are 5:6:7. Find the direction ratios of a line perpendicular to both these lines.

First write the numbers in this array:

$$\begin{matrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{matrix}$$

Then cross-multiply as explained above.

$$(3)(7) - (4)(6) = -3$$
$$(4)(5) - (2)(7) = 6$$
$$(2)(6) - (5)(3) = -3$$

The required direction ratios are $-3:6:-3$, which can be reduced by dividing by -3 to give $1:-2:1$.

After finding the direction ratios of a line perpendicular to two lines it is essential to check the result. To do this we must prove that the resulting line is actually perpendicular to each of the two given lines. This can be done as follows:

$$(1)(2) + (-2)(3) + (4)(1) = 0,$$
$$(1)(5) + (-2)(6) + (7)(1) = 0.$$

This check depends upon the fact that if two lines are perpendicular, the cosine of the angle between them is zero.

Example 2. A plane is determined by the three points $A(3, 4, 1)$, $B(5, 10, 16)$, $C(-3, 8, 22)$. Find the direction ratios of a line normal to this plane.

The direction ratios of line AB are

$$5 - 3:10 - 4:16 - 1 \quad \text{or} \quad 2:6:15.$$

The direction ratios of line AC are

$$-3 - 3:8 - 4:22 - 1 \quad \text{or} \quad -6:4:21.$$

Write these two sets of direction ratios in the form

$$\begin{array}{c} 2:6:15 \\ -6:4:21. \end{array}$$

Cross-multiply in the manner described above:

$$\begin{aligned} (6)(21) - (15)(4) &= 126 - 60 = 66. \\ (15)(-6) - (2)(21) &= -90 - 42 = -132. \\ (2)(4) - (-6)(6) &= 8 + 36 = 44. \end{aligned}$$

Therefore the direction ratios of a normal to the given plane are $66:-132:44$, which can be reduced by dividing by 44 to give $1.5:-3:1$.

Example 3. A certain plane is determined by the point $A(3, 2, 1)$ and the line $B(6, 2, 4)$ $C(1, 0, 5)$. Find the direction ratios of a normal to this plane.

The points A, B and A, C determine two lines that lie in the given plane. Their direction ratios are

$$\begin{array}{llll} AB & 6 - 3:2 - 2:4 - 1 & \text{or} & 3:0:3, \\ AC & 1 - 3:0 - 2:5 - 1 & \text{or} & -2:-2:4. \end{array}$$

These two sets of direction ratios can be reduced by dividing by 3 and -2, respectively, to give $1:0:1$ and $1:1:-2$. Write these ratios in the form

$$\begin{array}{c} 1:0:1 \\ 1:1:-2. \end{array}$$

Cross-multiply:

$$
\begin{aligned}
(0)(-2) - (1)(1) &= -1. \\
(1)(1) - (1)(-2) &= 3. \\
(1)(1) - (1)(0) &= 1.
\end{aligned}
$$

Therefore the direction ratios of a normal to the given plane are $-1:3:1$.

Example 4. The plane in which three main bearings of the landing gear must operate is determined by the three points $A(50, 3, 15)$, $B(52, 17, 8)$, $C(51, 22, 16)$. Find the direction ratios of a normal to this plane.

The direction ratios of AB are

$$52 - 50:17 - 3:8 - 15 \qquad \text{or} \qquad 2:14:-7.$$

The direction ratios of AC are

$$51 - 50:22 - 3:16 - 15 \qquad\qquad 1:19:1.$$

Write these ratios in the form

$$
\begin{aligned}
&2:14:-7 \\
&1:19:1.
\end{aligned}
$$

Cross-multiply:

$$
\begin{aligned}
(14)(1) - (19)(-7) &\quad 147. \\
(-7)(1) - (2)(1) &= -9 \\
(2)(19) - (1)(14) &= 24.
\end{aligned}
$$

Therefore the direction ratios of a normal to the given plane are $147:-9:24$, which can be reduced by dividing by -9 to give $-16\frac{1}{3}:1:-2\frac{2}{3}$.

5.7. True angle between a line and a plane, when the plane is determined by three points. When the given plane is a single-canted plane, *i.e.*, when it is on edge in one view, as in the case of the canted bulkhead in Fig. 5.4 of Art. 5.4, the direction ratios of a normal to the plane are easy to calculate. Sometimes, however, the given plane is determined by three points. This is often true in the case of a double-canted plane, which is not on edge in any of the three basic orthographic views. In this case we can calculate the direction ratios of a normal to the plane by the method described in the preceding article.

Example 1. A certain plane is determined by the three points $A(15, 85, 16)$, $B(13, 82, 12)$, and $C(10, 79, 9)$. A certain line is determined by the two points $D(17, 5, 6)$ and $E(16, 5, 2)$. Find the true angle between the line DE and the plane ABC.

Select any two lines in the plane ABC, such as BA and CA. Calculate their direction ratios. They are

$$
\begin{aligned}
&BA \quad 15 - 13:85 - 82:16 - 12 \qquad \text{or} \qquad 2:3:4, \\
&CA \quad 15 - 10:85 - 79:16 - 9 \qquad \text{or} \qquad 5:6:7.
\end{aligned}
$$

Cross-multiply as explained in Example 1, Art. 5.6. We obtain finally $1:-2:1$. These are the direction ratios of a line perpendicular to AB and AC, and therefore perpendicular to the plane ABC. Calculate the direction ratios of ED. They are

$$ED \quad 17 - 16:5 - 5:6 - 2 \qquad 1:0:4.$$

Therefore the true angle between the line DE and a normal to the plane ABC is given by

$$\cos\theta \qquad \frac{(1)(1) + (-2)(0) + (1)(4)}{\sqrt{(1)^2 + (-2)^2 + (1)^2}\,\sqrt{(1)^2 + (0)^2 + (4)^2}}$$

$$\cos\theta \qquad 5$$

$$\cos\theta \qquad 10.0995 \qquad 0.49507.$$

$$\theta = 60°19'32''.$$

This is the complement of the true angle between the line DE and the plane ABC, and the true angle between the line and the plane is therefore $29°40'28''$.

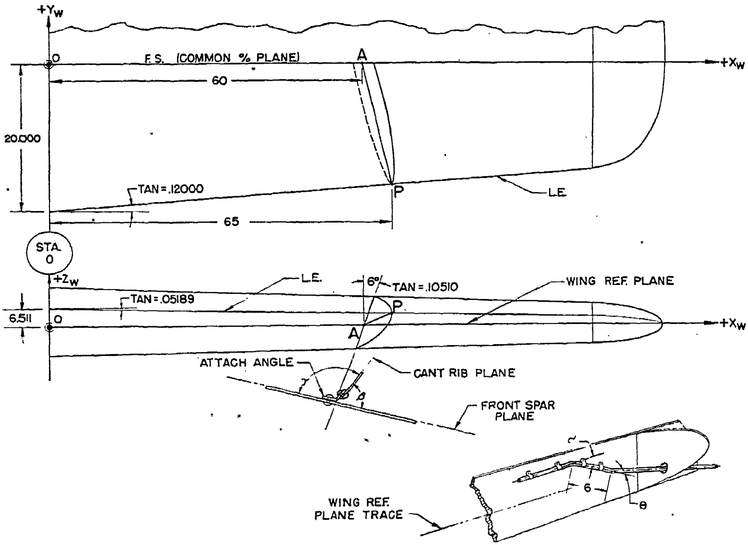


FIG. 5.7.

Example 2. On the outboard side of a deicer duct lies a cant rib plane which is defined by its intersection with the front spar plane and its intersection with the leading edge of the wing (see Fig. 5.7). The pressure tube

leading in from the outer wing Pitot tube runs through the cant rib plane with given direction ratios of $1:-1:-0.5$ to strike the front spar plane, makes a single angle bend for 6 in., then bends again to run parallel to the wing reference plane trace on the front spar plane. What are the two bend angles required to make the tube?

A line normal to the front spar plane has direction ratios or direction cosines of $0:1:0$. The direction ratios of the pressure tube are $1:-1:-0.5$. Its direction cosines are $\frac{1}{1.5}, -\frac{1}{1.5}, -\frac{0.5}{1.5}$. The true angle between the front spar plane and the pressure tube is given by

$$\cos \phi \quad [illegible] -\frac{[illegible]}{1.5} = -0.66667.$$

Notice that $\phi > 90°$, since the cosine is negative.

To find the actual angle of bend,

$$\sin \theta = 0.66667.$$
$$\theta = 41°48'38''.$$

The other angle of bend made in the front spar plane is found by the following procedure:

The direction cosines of the wing reference plane trace on the front spar plane are 1, 0, 0. The direction ratios of the 6-in. piece of pressure tube in the front spar plane are $1:0:-0.5$.

$$[illegible] \frac{[illegible]}{1.11803} = 0.89443$$
$$\alpha = 26°33'51''.$$

This is the angle between the 6-in. piece of pressure tube and the x_w axis.

These are the two angles (θ and α) required to prefabricate the pressure tube.

5.8. True angle between two planes, when the planes are each determined by three points. Consider the problem of determining the true angle between two planes, when each of the planes is determined by three points. Briefly, the method consists of calculating the direction ratios of a normal to each of the two given planes and then finding the true angle between these two normals. This angle is equal to the true angle between the two given planes. The procedure will be made clear in the following example.

Example 1. Find the true angle between the plane $A(15, 85, 16)$ $B(13, 82, 12)$ $C(10, 79, 9)$ and the plane $D(26, 12, 17)$ $E(24, 11, 17)$ $F(23, 10, 15)$.

The direction ratios of BA are $2:3:4$ and the direction ratios of CA are $5:6:7$. The direction ratios of a normal to ABC are $1:-2:1$ (see Example 1, Art. 5.6).

The direction ratios of ED are $2:1:0$ and the direction ratios of FD are $3:2:2$. The direction ratios of a normal to DEF are calculated by cross-multiplying

$$\begin{array}{c} 2:1:0 \\ 3:2:2. \end{array}$$

The results are

$$\begin{aligned} (1)(2) - (2)(0) &= 2, \\ (0)(3) - (2)(2) &= -4, \\ (2)(2) - (3)(1) &= 1. \end{aligned}$$

The direction ratios of a normal to DEF are therefore $2:-4:1$.

The true angle between the normal to ABC and the normal to DEF is given by

$$\cos\theta \quad \frac{(1)(2) + (-2)(-4) + (1)(1)}{\sqrt{1^2 + (-2)^2 + 1^2}\sqrt{(2)^2 + (-4)^2 + 1^2}}.$$

$$\cos\theta \quad 11$$

$$\cos\theta \quad \frac{11}{11.22497} \quad 0.97996.$$

$$11°29'20''.$$

Example 2. Referring to Example 2 of Art. 5.7 and Fig. 5.7, what is the angle required to prefabricate the attach angle that fastens the cant rib to the front spar?

The coordinates of point P are $x_w = 65$, $y_w = -12.200$, and $z_w = 3.138$ as calculated from Fig. 5.7. The coordinates of point A are $x_w = 60$, $y_w = 0$, $z_w = 0$. The direction ratios of line AP are $5:-12.200:3.138$, or, simplifying, $-0.40984:1:-0.25721$. The direction ratios of the line of intersection of the cant rib plane with the front spar plane as calculated from Fig. 5.7 are $0.10510:0:1$. The direction ratios of a line normal to the cant rib plane can be calculated by cross-multiplying

$$\begin{array}{c} -0.40984:1:-0.25721 \\ 0.10510:0:1. \end{array}$$

The results are $1:0.38281:-0.10510$.

The angle of the attach angle is the true angle between the front spar plane and the cant rib plane, the normals to which are, respectively, $0:1:0$ and $1:0.38281:-0.10510$.

$$\frac{(0)(1) + (1)(0.38281) + (0)(-0.10510)}{\sqrt{(0)^2 + (1)^2 + (0)^2}\sqrt{(1)^2 + (0.38281)^2 + (0.10510)^2}} \quad \frac{0.38281}{1.07591} = 0.35580.$$

$$\beta = 69°9'27''.$$

The angle as shown on the drawing of the attach angle would be the supplement of β, or $\gamma = 110°50'33''$.

CHAPTER 6

EQUATIONS OF PLANES

In this chapter we shall show how to write the equation of a plane. It is advisable, in applying solid analytic geometry to the airplane, to write the equations of the basic planes and to tabulate them for future reference. This will result in a saving of time and energy. The basic planes are used so often in problems that it would be a duplication of effort to have to calculate their equations each time a new problem arises. We shall consider certain other topics in this chapter, such as the distance from the origin to a plane, the distance from a point to a plane, parallel and perpendicular planes, the line of intersection of two planes, the point where a line pierces a plane, and the angle made on a plane by its intersections with two other planes.

Once the equation of a plane has been written, the direction ratios of a line normal to the plane are immediately available for use in finding the true angle between a line and a plane and the true angle between two planes.

6.1. Normal form of the equation of a plane. Consider a plane ABC, as shown in Fig. 6.1. Let ON be the perpendicular from the origin to the plane. Let α, β, and γ be the direction angles of the line ON. Let D be the point where ON intersects the plane ABC, and let p be the length of the segment OD. Let $P(x, y, z)$ be any point on the plane ABC. The equation of the plane ABC is $x \cos \alpha + y \cos \beta + z \cos \gamma = p$.

To verify this equation, notice that DP is perpendicular to OD because OD is perpendicular to the plane ABC, and DP lies in the plane ABC. Now the projection of OP on ON is OD, since angle ODP is a right angle. Also, the projection of OP on ON is equal to the sum of the projections of the line segments representing the x, y, z coordinates of P. Therefore, equating these two expressions for the projection of OP on ON,

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

This relation is true for any point P that lies in the plane ABC. It is not true for any point not in the plane ABC, because for such a point the angle ODP would not be a right angle. Since

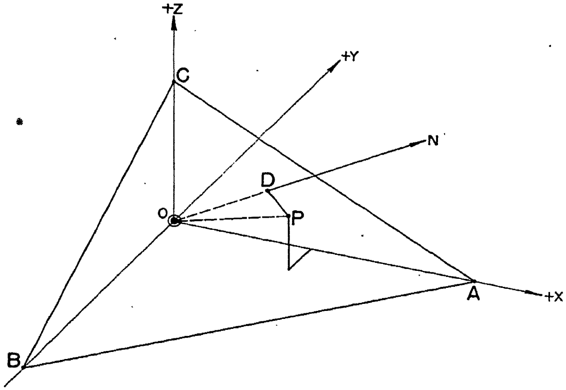


FIG. 6.1.

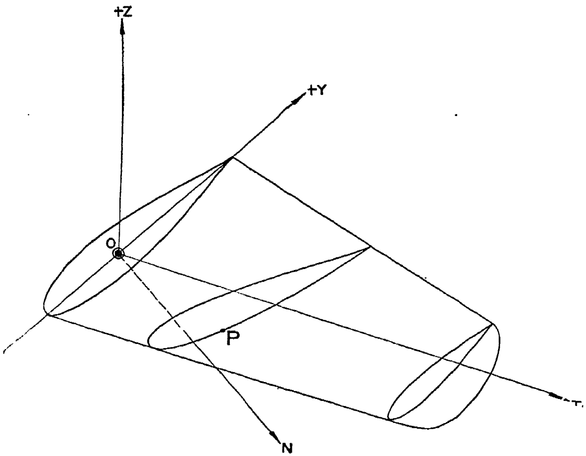


FIG. 6.2.

the equation is satisfied by all points in the plane and by no points not in the plane, it is the equation of the plane. This is called the *normal* form of the equation of a plane, because $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are the direction cosines of a normal to the plane.

Example 1. If the direction cosines of a normal to a certain plane are $\frac{2}{7}$, $\frac{6}{7}$, and $\frac{3}{7}$, and the perpendicular distance from the origin to the plane is 13 in., write the equation of the plane.

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$
$$\cos \alpha = \tfrac{2}{7}, \qquad \cos \beta = \tfrac{6}{7}, \qquad \cos \gamma = \tfrac{3}{7}.$$
$$p = 13.$$
$$\tfrac{2}{7}x + \tfrac{6}{7}y + \tfrac{3}{7}z = 13.$$

Example 2. The direction ratios of a normal to a double-canted (skewed) wing rib are $1:-0.03491:0.12187$. A point on the rib is (100.127, 40.039, -10.426). Find the equation of the plane of the rib in its normal form (see Fig. 6.2).

First calculate the value of $\sqrt{1^2 + (-0.03491)^2 + (0.12187)^2}$. This is 1.00800. Then the direction cosines of the normal to the rib are

$$\begin{aligned} & \qquad\quad 0.99206. \\ & 1.00800 \\ \cos \beta \quad & \frac{-0.03491}{1.00800} \quad -0.03463. \\ \cos \gamma \quad & \frac{0.12187}{1.00800} = 0.12090. \end{aligned}$$

The projection of OP on ON is equal to the perpendicular distance from the origin to the plane. This distance is therefore

$$(100.127)(0.99206) + (40.039)(-0.03463) + (-10.426)(0.12090) = 96.685.$$

Therefore the equation of the plane of the rib is

$$0.99206x - 0.03463y + 0.12090z = 96.685.$$

Notice that the coefficients of x, y, and z are the direction cosines of a normal to the rib and the constant term on the right side of the equation is the perpendicular distance from the origin to the plane of the rib.

The equation of a plane is always a linear equation of the type

$$Ax + By + Cz + D = 0$$

and conversely, every equation of this type represents a plane. It follows, therefore, that the equation of a plane is not always in the normal form. For example, the equation

$$0.99206x - 0.03463y + 0.12090z = 96.685$$

is in the normal form, as explained above, but the equation

$$2(0.99206)x - 2(0.03463)y + 2(0.12090)z = 2(96.685),$$

which represents the same plane, is not in the normal form, because the coefficients of x, y, z are now 1.98412, -0.06926,

0.24180 and these numbers do not constitute a set of direction cosines, since the sum of their squares is not equal to one. However, the equation $1.98412x - 0.06926y + 0.24180z = 193.370$ can be put into the normal form by dividing through by

$$\sqrt{(1.98412)^2 + (-0.06926)^2 + (0.24180)^2}.$$

The result is

$$0.99206x - 0.03463y + 0.12090z = 96.685,$$

which is in the normal form. Now the coefficients of x, y, and z are the direction cosines of a normal to the plane, and the perpendicular distance from the origin to the plane is 96.685.

Example 3. Determine whether the equation

$$0.96643x - 0.22303y + 0.12755z = 10$$

is in the normal form.

Calculate the value of $\sqrt{(0.96643)^2 + (-0.22303)^2 + (0.12755)^2}$. The result is one. Therefore the equation is in the normal form. A consequence of this fact is that 10 is the perpendicular distance from the origin to the plane.

Example 4. Determine whether the equation $2x - 3y + 4z = 6$ is in the normal form.

The coefficients are each larger than unity (one), so the sum of their squares is obviously larger than unity. The equation is not in the normal form.

Example 5. Reduce the equation

$$1.23488x - 0.27046y + 1.02116z = 13.049$$

to the normal form.

Calculate the value of $\sqrt{(1.23488)^2 + (-0.27046)^2 + (1.02116)^2}$. The result is 1.62507. Divide each term of the equation by this number. The result is $0.75989x - 0.16643y + 0.62838z = 8.030$. This is the normal form of the given equation. The coefficients of x, y, and z are the direction cosines of a normal to the plane, and 8.030 is the perpendicular distance from the origin to the plane.

Example 6. Reduce the equation

$$0.03489x + 0.86048y + 0.99406z = 29.642$$

to the normal form.

Calculate the value of $\sqrt{(0.03489)^2 + (0.86048)^2 + (0.99406)^2}$. The result is 1.31522. Divide each term of the equation by this number. The result is

$$0.02653x + 0.65425y + 0.75581z = 22.538.$$

This is the normal form of the equation. The coefficients of x, y, and z are the direction cosines of a normal to the plane, and the perpendicular distance from the origin to the plane is 22.538.

6.2. General form of the equation of a plane. The equation of a plane in space is usually written in the form

$$Ax + By + Cz + D = 0.$$

All points whose coordinates satisfy the equation lie in the plane, and no point whose coordinates do not satisfy the equation can lie in the plane. In this sense the equation represents the plane.

The equation of a plane perpendicular to the xy plane will have the term in z missing:

$$Ax + By + D = 0.$$

The equation of a plane perpendicular to the xz plane will have the term in y missing:

$$Ax + Cz + D = 0.$$

The equation of a plane perpendicular to the yz plane will have the term in x missing:

$$By + Cz + D = 0.$$

For example, the plane $2x + 3y + 7 = 0$ is perpendicular to the xy plane, the plane $4x + 5z + 2 = 0$ is perpendicular to the xz plane, and the plane $3y - 8z + 4 = 0$ is perpendicular to the yz plane.

An example of such a plane is the plane of the canted bulkhead in Fig. 5.4. When a plane is perpendicular to one of the three reference planes it will appear on edge in one of the three basic orthographic views. Therefore when a plane is on edge in one view its equation will have one unknown missing. It is especially easy to write the equation of such a plane. The edge view of the plane can be considered as a line, and the equation of the line as determined in plane analytic geometry will be the equation of the plane.

Example 1. Write the equation of the vertical rib at wing station 0 and the equation of the vertical rib at wing station 150 in Fig. 6.3.

The planes of the vertical ribs being normal to the x_wz_w plane, their equations will contain terms in x_w and z_w, but not y_w. Treating the vertical rib at station 0 as a line, its equation is $z_w = x_w \tan 86°$, since it goes through

the origin and has a slope equal to tan 86°. Therefore its equation is $z_w = 14.301x_w$. The vertical rib at station 150, when treated as a line,

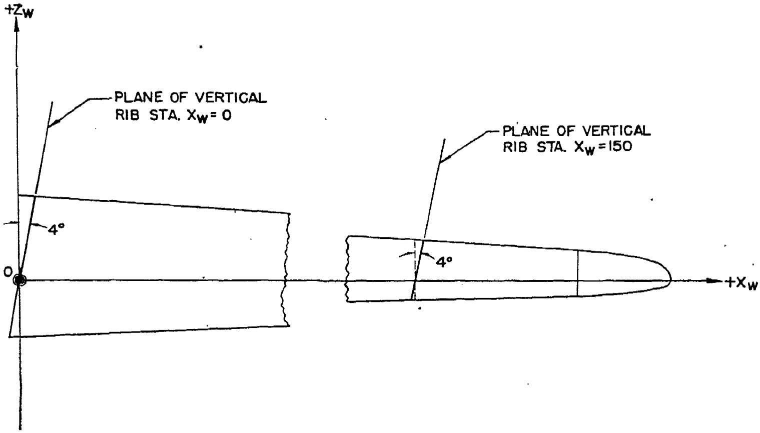


FIG. 6.3.

goes through the point (150, 0) and has a slope of tan 86°. Therefore its equation is obtained by using the point-slope equation of a line.

$$z_w - z_1 = m(x_w - x_1).$$
$$z_w - 0 = 14.301(x_w - 150).$$
$$z_w = 14.301x_w - 2145.15.$$

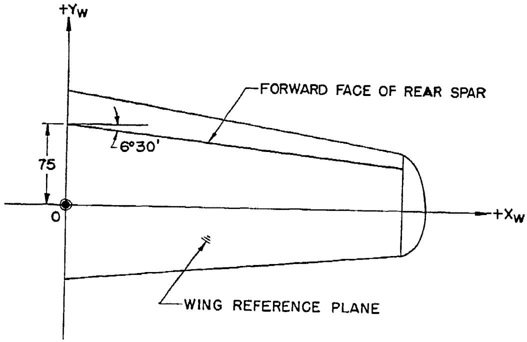


FIG. 6.4.

Example 2. Write the equation of the plane of the forward face of the rear spar in Fig. 6.4.

In this figure the spar is designed to be normal to the wing reference plane, which is the $x_w y_w$ plane. Therefore its equation will contain x_w and y_w but not z_w. Since it is on edge in this view it can be treated as a line. It goes through the point (0, 75) and has a slope of tan (180° − 6°30′), or tan 173°30′, which is −0.11394. Therefore its equation is obtained from the point-slope equation of a line:

$$\begin{aligned} y_w - y_1 &= m(x_w - x_1). \\ y_w - 75 &= -0.11394(x_w - 0). \\ y_w &= -0.11394x_w + 75. \end{aligned}$$

The equation of a plane that is parallel to a basic reference plane contains only one variable. The equation of a plane parallel to the xy plane is $z = D$. The equation of a plane parallel to the xz plane is $y = D$. The equation of a plane parallel to the yz plane is $x = D$. In these three equations D is any real number.

Examples of planes parallel to basic reference planes are buttock line planes, fuselage station planes, and water line planes in rigged position. The equations are

Buttock line planes,	$x = D$.
Fuselage station planes,	$y = D$.
Water line planes,	$z = D$.

In a wing, the normal ribs are normal to the wing reference plane (or wing chord plane) and have equations of the type $x_w = D$, since they are parallel to the $y_w z_w$ plane.

6.3. Equation of a plane determined by a point and the direction ratios of a normal to the plane. If the direction ratios of a normal to a plane are $a:b:c$ and a point on the plane is

$$P_1(x_1, y_1, z_1),$$

then the equation of the plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This equation can be derived as follows: Let $P(x, y, z)$ be any point on the required plane. Then PP_1 is any line through P_1 and lies in the plane. The direction ratios of PP_1 are

$$x - x_1 : y - y_1 : z - z_1.$$

The normal to the plane is perpendicular to PP_1; so the sum of the products of the corresponding direction ratios of the normal and PP_1 must equal zero, *i.e.*,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is the equation of the plane, because it is true for all points on the plane, and for no point not on the plane.

Example 1. A certain plane is determined by the point (3, 2, 5) and a normal to the plane whose direction ratios are 7:8:9. Write the equation of the plane.

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$
$$7(x - 3) + 8(y - 2) + 9(z - 5) = 0.$$

Example 2. A plane is determined by the point (3, 2, 5) and a normal to the plane whose direction cosines are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$. Write the equation of the plane.

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$
$$\tfrac{1}{3}(x - 3) + \tfrac{2}{3}(y - 2) + \tfrac{2}{3}(z - 5) = 0.$$

Example 3. The direction ratios of a line normal to a given plane are $1:-0.42969:0.59169$, and a point on the plane is (32.576, −7.324, 10.453). Write the equation of the plane.

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$
$$1(x - 32.576) - 0.42969(y + 7.324) + 0.59169(z - 10.453) = 0.$$

This equation can be reduced to the general form by multiplying and collecting terms to give

$$x - 0.42969y + 0.59169z = 41.908.$$

This result can be reduced to the normal form, if desired, by dividing by the square root of the sum of the squares of the coefficients of x, y, and z to give

$$0.80721x - 0.34685y + 0.47762z = 33.828.$$

Example 4. From the information in Fig. 6.5 find the equation of the plane of the nose rib, which is normal to the leading edge in the plan view.

This is a chord plane wing, and the leading edge is therefore in the plane of the paper. The leading edge is normal to the required plane of the nose rib. The direction ratios of the leading edge can be determined as follows: The coordinates of O are (0, 0, 0). The coordinates of A are (cos 5°, sin 5°, 0). The direction cosines of OA are therefore

$$\cos 5°, \sin 5°, 0.$$

The point P lies on the plane of the nose rib, and its coordinates are (200, 0, 0). Therefore the equation of the plane of the nose rib is

$$0.99619(x_w - 200) + 0.08716(y_w - 0) + 0(z_w - 0) = 0,$$

which can be reduced by multiplying,

$$0.99619x_w + 0.08716y_w - 199.238 = 0.$$

Example 5. The direction ratios of a normal to a double-skewed wing rib are $1:-0.03491:0.12187$, and a point on the rib is (100.127, 40.039, -10.426). Find the equation of the plane of the rib.

$$a(x_w - x_1) + b(y_w - y_1) + c(z_w - z_1) = 0.$$
$$1(x_w - 100.127) - 0.03491(y_w - 40.039) + 0.12187(z_w + 10.426) = 0.$$

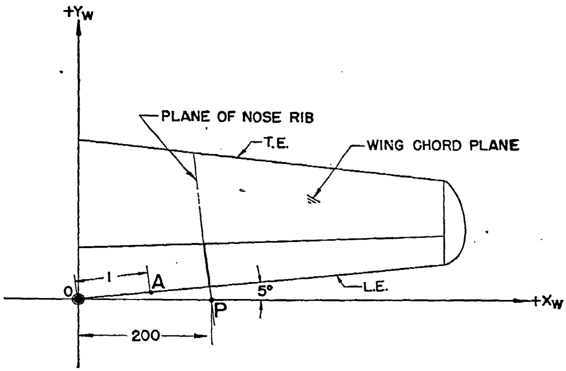


FIG. 6.5.

This result can be reduced to the general form by multiplying and collecting terms to give

$$x_w - 0.03491y_w + 0.12187z_w = 97.459.$$

This equation can be reduced to the normal form, if desired, by dividing by

$$\sqrt{1^2 + (-0.03491)^2 + (0.12187)^2} = 1.00800.$$

The normal form of the equation is therefore

$$0.99206x_w - 0.03463y_w + 0.12090z_w = 96.685.$$

A different solution of this problem was given in Example 2, Art. 6.1.

Example 6. A plane is determined by the three points $A(3, 2, 1)$, $B(6, 2, 4)$, $C(1, 0, 5)$. Find the equation of the plane.

The direction ratios of AB are $3:0:3$. The direction ratios of BC are $-5:-2:1$. The direction ratios of a line normal to AB and BC are obtained by cross-multiplying

$$\begin{array}{r} 3:\ \ 0:3 \\ -5:-2:1. \end{array}$$

The results are $6:-18:-6$, which can be reduced by dividing by 6 to give $1:-3:-1$. A normal to AB and BC will also be normal to the plane ABC. The equation of the plane ABC is therefore

$$1(x-3) - 3(y-2) - 1(z-1) = 0.$$
$$x - 3y - z + 4 = 0.$$

The method described in Example 6 is applicable when the given plane is determined by three points. The procedure is as follows:

1. Find the direction ratios of a line determined by two of the three given points.
2. Find the direction ratios of a different line determined by two of the three given points.
3. Find the direction ratios of a normal to these two lines by cross-multiplying the direction ratios obtained in steps 1 and 2.
4. Write the equation of the plane in the form

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0,$$

where $a:b:c$ are the direction ratios of a normal to the plane, and (x_1, y_1, z_1) are the coordinates of any one of the three given points.

6.4. Distance from a point to a plane. The positive direction on a line perpendicular to a plane is assumed to agree with the direction on a line drawn from the origin perpendicular to the plane.

Example 1. The positive direction on a line perpendicular to the plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p$$

is $+:+:+$. The positive direction on a line perpendicular to the plane

$$x \cos \alpha - y \cos \beta + z \cos \gamma = p$$

is $+:-:+$. The positive direction on a line perpendicular to the plane

$$x \cos \alpha + y \cos \beta - z \cos \gamma = -p$$

is the same as the positive direction on a line perpendicular to

$$-x \cos \alpha - y \cos \beta + z \cos \gamma = p,$$

which is obtained from the preceding equation by multiplying by -1, so the positive direction is $-:-:+$. We multiply by -1 in order to make p positive.

The distance from a point to a plane is positive if the point and the origin lie on opposite sides of the plane. The distance from a point to a plane is negative if the point and the origin lie on the same side of the plane.

The distance from a point $P_1(x_1, y_1, z_1)$ to the plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p$$

is given by

$$= x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p.$$

This formula can be derived as follows (see Fig. 6.6):

The projection of OP_1 on ON is equal to $p + d$. The projection of OP_1 on ON is equal to the sum of the projections of OA, AB, and BP_1 on ON, since OP_1 is the closing line segment for

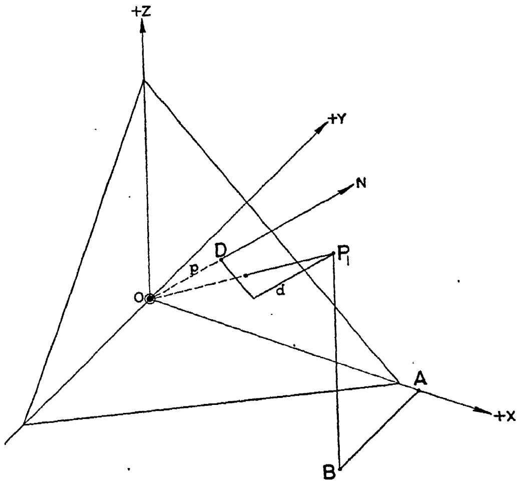


FIG. 6.6.

the broken line segment $OABP_1$. Now the projection of OA on ON is $x_1 \cos \alpha$, the projection of AB on ON is $y_1 \cos \beta$, and the projection of BP_1 on ON is $z_1 \cos \gamma$, where $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are the direction cosines of ON.

Therefore

$$p + d = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma,$$

and finally

$$d = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p.$$

Example 2. Find the perpendicular distance from the point (6, 9, 12) to the plane $\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z = 7$.

The equation of the plane is in the normal form, since

$$(\tfrac{1}{3})^2 + (\tfrac{2}{3})^2 + (\tfrac{2}{3})^2 = 1.$$

Therefore the distance from the point to the plane is

$$d = (\tfrac{1}{3})(6) + (\tfrac{2}{3})(9) + (\tfrac{2}{3})(12) - 7.$$
$$d = 2 + 6 + 8 - 7.$$
$$d = 9.$$

Example 3. Find the perpendicular distance from a point on the center line of a landing gear fitting (89.798, −20.026, −7.605) to a double-skewed wing rib whose equation is $0.99206x - 0.03463y + 0.12090z = 96.685$.

The equation of the plane is in the normal form, since

$$(0.99206)^2 + (-0.03463)^2 + (0.12090)^2 = 1.$$

Therefore the distance from the point to the plane is

$$d = (0.99206)(89.798) + (-0.03463)(-20.026) + (0.12090)(-7.605) - 96.685.$$
$$d = -7.826.$$

Since the answer is negative, the given point and the origin are on the same side of the plane.

Example 4. Find the perpendicular distance from the point (6, 9, 12) to the plane $x + 2y + 2z = 21$.

The equation of the plane is not in the normal form, since $1^2 + 2^2 + 2^2 \neq 1$. We reduce it to the normal form by dividing by $\sqrt{1^2 + 2^2 + 2^2}$, which is equal to 3. We obtain

$$\tfrac{1}{3}x + \tfrac{2}{3}y + \tfrac{2}{3}z = 7.$$

Now the problem reduces to Example 2 in this article.

Example 5. Find the perpendicular distance from the point (89.798, −20.026, −7.605) to the plane

$$0.49603x - 0.01732y + 0.06045z = 48.342.$$

The equation of the plane is not in the normal form, since

$$(0.49603)^2 + (-0.01732)^2 + (0.06045)^2 \neq 1.$$

We reduce it to the normal form by dividing by

$$\sqrt{(0.49603)^2 + (-0.01732)^2 + (0.06045)^2},$$

which is equal to 0.50000. We obtain

$$0.99206x - 0.03464y + 0.12090z = 96.684.$$

Now the problem reduces to Example 3 in this article.

The distance from a point to a plane is one of the fundamental concepts of both descriptive geometry and solid analytic geome-

try. We have shown a mathematical method for calculating this distance which can be made as accurate as the given dimensions. This is an extremely useful method and finds many applications which arise frequently in the designing, engineering, tooling, lofting, and jig-building departments. The same given information needed to solve the problem mathematically is needed to solve the problem by descriptive geometry layout methods. The mathematical method described in this article is perfectly general and applies equally well to single-canted and double-canted planes.

6.5. Parallel planes and perpendicular planes. When two planes are parallel to each other they are perpendicular to the same line. Therefore two planes are parallel if and only if the coefficients of x, y, z in their equations are equal or proportional. For example, the two planes

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0 \\ A_2x + B_2y + C_2z + D_2 &= 0 \end{aligned}$$

are parallel if and only if

$$\frac{A_1}{A_2} \quad \frac{B_1}{B_2} \quad \frac{C_1}{C_2}.$$

Example 1. Are the following planes parallel?

$$\begin{aligned} 2x + 3y + 8z &= 6. \\ 4x + 6y + 16z &= 7. \end{aligned}$$

The planes are parallel, because

$$\tfrac{2}{4} = \tfrac{3}{6} = \tfrac{8}{16}.$$

If the equations of the parallel planes are in normal form, then the corresponding coefficients will be equal or opposite in sign. Suppose that the equations of two parallel planes are

$$\begin{aligned} x \cos \alpha_1 + y \cos \beta_1 + z \cos \gamma_1 &= p_1, \\ x \cos \alpha_2 + y \cos \beta_2 + z \cos \gamma_2 &= p_2. \end{aligned}$$

Then $\cos \alpha_1 = \quad \cos \alpha_2$; $\quad \cos \beta_1 = \quad \cos \beta_2$;

$\cos \gamma_1 = \quad \cos \gamma_2$,

or $\quad \cos \alpha_1 = -\cos \alpha_2$; $\quad \cos \beta_1 = -\cos \beta_2$;

$\cos \gamma_1 = -\cos \gamma_2$.

For example, if the equations of two planes in the normal form are

$$\begin{aligned} \tfrac{1}{3}x + \tfrac{2}{3}y + \tfrac{2}{3}z &= 7, \\ -\tfrac{1}{3}x - \tfrac{2}{3}y - \tfrac{2}{3}z &= -8, \end{aligned}$$

these planes are parallel.

Since p_1 is the distance from the origin to the first plane and p_2 is the distance from the origin to the second plane, then $p_1 - p_2$ is the distance between the two parallel planes.

If the coefficients of x, y, z are equal, then the algebraic difference $p_1 - p_2$ is the distance between the two planes. If the coefficients of x, y, z are equal numerically but opposite in sign, then it is necessary to multiply one of the equations by minus one before taking the algebraic difference $p_1 - p_2$.

Example 2. Determine whether the following two planes are parallel:

$$\begin{aligned} 0.09438x + 0.54378y - 0.83390z &= 10.091, \\ 0.54969x + 0.44968y - 0.70400z &= 21.568. \end{aligned}$$

Calculate the values of

$$\frac{0.09438}{0.54969}, \quad \frac{0.54378}{0.44968}, \quad \frac{-0.83390}{-0.70400}.$$

The quotient in each case is different, and so the two given planes are not parallel.

Example 3. Find the distance between the parallel planes

$$\begin{aligned} 0.09438x + 0.54378y - 0.83390z &= 10.091, \\ -0.09438x - 0.54378y + 0.83390z &= 21.568. \end{aligned}$$

Squaring and adding the coefficients of x, y, z in each of the two equations, we obtain one (1) in each case. The equations are in the normal form. The coefficients are equal numerically but are opposite in sign. Multiply the second equation by -1. The planes are parallel, and the distance between them is $10.091 - (-21.568) = 31.659$.

Consider the two planes

$$\begin{aligned} A_1x + B_1y + C_1z &= D_1, \\ A_2x + B_2y + C_2z &= D_2. \end{aligned}$$

The coefficients of x, y, and z are direction ratios of normals to the planes. Therefore the two planes are perpendicular if and only if

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

Example 4. Are the planes

$$\begin{aligned} 2x - y - 4z &= 6 \\ -x + 2y - z &= 7 \end{aligned}$$

perpendicular?

The planes are perpendicular, because

$$(2)(-1) + (-1)(2) + (-4)(-1) = 0.$$

Example 5. Are the planes

$$\begin{aligned} 3x + y + 2z &= 5 \\ x + 5y + 3z &= 7 \end{aligned}$$

perpendicular?

The planes are not perpendicular, because

$$(3)(1) + (1)(5) + (2)(3) \neq 0.$$

6.6. The intercept form of the equation of a plane. If a given plane intersects the x axis a units from the origin, the y axis b units from the origin, and the z axis c units from the origin, then its equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

The numbers a, b, and c are intercepts of the plane on the x axis, y axis, and z axis, respectively. This equation can be derived as follows. The equation of any plane is

$$Ax + By + Cz + D = 0.$$

Let $y = 0$ and $z = 0$. Then $Ax + D = 0$ and $x = -\dfrac{D}{A}\cdot$ Let $x = 0$ and $z = 0$. Then $By + D = 0$ and $y = -\dfrac{D}{B}\cdot$ Let $\quad 0$ and $y = 0$. Then $Cz + D = 0$ and $z = -\dfrac{D}{C}\cdot$ The intercepts are therefore $-\dfrac{D}{A}$, $-\dfrac{D}{B}$, and $-\dfrac{D}{C}\cdot$ These fractions may be equated to a, b, and c, respectively, and so

$$A = -\frac{D}{a}, \qquad B = -\frac{D}{b}, \qquad C = -\frac{D}{c}.$$

Substituting these values for A, B, and C in the equation

$$Ax + By + Cz + D = 0,$$

we obtain

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0,$$

which can be reduced by dividing by $-D$ to give

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Example 1. Determine the equation of a plane, the intercepts of which are 2, 3, and 4.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1.$$

This equation can be reduced to a form with integral coefficients, if desired, by multiplying by 12. The result is

$$6x + 4y + 3z = 12.$$

This equation can be reduced to the normal form, if desired, by dividing by $\sqrt{6^2 + 4^2 + 3^2}$, which is equal to $\sqrt{61}$. The result is

$$\frac{6}{\sqrt{61}}x + \frac{4}{\sqrt{61}}y + \frac{3}{\sqrt{61}}z - \frac{12}{\sqrt{61}}$$

Example 2. A skewed (canted) fuselage frame has intercepts 15.093, 10.968, and 20.314 on the x axis, y axis, and z axis, respectively. Write the equation of the plane of the frame.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

$$\frac{x}{15.093} + \frac{y}{10.968} + \frac{z}{20.314} = 1.$$

This result can be reduced to the normal form, as explained in Example 1. It is worth while to notice that the coefficients of x, y, and z are

15.093 10.968 and 20.314

and these are the direction ratios of a normal to the plane. By division they can be reduced to give

$$0.06626 : 0.09117 : 0.04923.$$

Example 3. A diagonal beam in the wing is determined by its intercepts on the x_w, y_w, and z_w axis, which are 40.029, 36.341, and 70.918, respectively. Find its equation.

$$\frac{x_w}{a} + \frac{y_w}{b} + \frac{z_w}{c} = 1.$$

$$\frac{x_w}{40.029} + \frac{y_w}{36.341} + \frac{\cdot}{70.918} = 1.$$

This result can be reduced to the normal form, as explained in Example 1.

6.7. Angle between a line and a plane, when the equation of the plane is given. Consider the problem of determining the true angle between a line and a plane when the equation of the plane is given.

Example 1. The equation of a certain plane in the normal form is

$$0.13459x + 0.97955y + 0.14955z = 33.838.$$

A certain line has direction cosines equal to 0, 0.99618, -0.08732. Find the true angle between the line and the plane.

Since the equation of the plane is in normal form, the direction cosines of a normal to the plane are 0.13459, 0.97955, 0.14955. The true angle between the given line and this normal is given by

$$\cos\theta = (0)(0.13459) + (0.99618)(0.97955) + (-0.08732)(0.14955).$$
$$\cos\theta = 0.96275.$$
$$\theta = 15°41'15''.$$

The true angle between the given line and the given plane is the complement of this angle, and so the required angle is 74°18′45″.

Example 2. The equation of a certain plane in the general form is

$$6x + 2y - 3z = 10.$$

A certain line is determined by the two points (16, 8, 12) and (13, 12, 12). Find the true angle between the given line and the given plane.

The direction ratios of a normal to the given plane are $6:2:-3$. The direction ratios of the given line are

$$16 - 13:8 - 12:12 - 12 \qquad \text{or} \qquad 3:-4:0.$$

The true angle between the normal and the given line is given by

$$\cos\theta = \frac{(6)(3) + (2)(-4) + (-3)(0)}{\sqrt{6^2 + 2^2 + (-3)^2}\sqrt{3^2 + (-4)^2 + 0^2}}.$$
$$\cos\theta = 0.28571.$$
$$\theta = 73°23'56''.$$

The true angle between the given line and the given plane is the complement of this angle. The required angle is therefore 16°36′4″.

Example 3. A certain wing is rigged by the wing reference plane system. A cant rib in the wing has the equation

$$x_w + 0.06993y_w = 52.013.$$

The direction ratios of the flap hinge center line in this wing are

$$1:0.09171:0.13843.$$

Find the true angle between the line and the plane.

The direction ratios of a normal to the plane of the cant rib are

$$1:0.06993:0.$$

The true angle between the flap hinge center line and the normal is given by

$$\cos\theta = \frac{(1)(1) + (0.09171)(0.06993) + (0.13843)(0)}{\sqrt{1^2 + (0.09171)^2 + (0.13843)^2}\sqrt{1^2 + (0.06993)^2 + 0^2}}.$$

$$\cos\theta = 0.99040.$$
$$\theta = 7°56'45''.$$

The true angle between the flap hinge center line and the plane of the cant rib is the complement of this angle. The required angle is therefore 82°3′15″.

Example 4. The equation of a certain plane is

$$0.02498x - 0.02752y + 0.01410z = 1.$$

A certain line is determined by the two points (36.025, 15.175, 18.250) and (17.512, 5.068, 12.125). Find the true angle between the line and the plane.

The direction ratios of a normal to the given plane are

$$0.02498:-0.02752:0.01410.$$

The direction ratios of the given line are

$$36.025 - 17.512:15.175 - 5.068:18.250 - 12.125$$

or

$$18.513:10.107:6.125.$$

The true angle between the normal and the given line is given by

$$\cos\theta = \frac{(0.02498)(18.513) + (-0.02752)(10.107) + (0.01410)(6.125)}{\sqrt{(0.02498)^2+(-0.02752)^2+(0.01410)^2}\sqrt{(18.513)^2+(10.107)^2+(6.125)^2}}.$$

$$\cos\theta = 0.31002.$$
$$\theta = 71°56'22''.$$

The true angle between the given line and the given plane is the complement of this angle. The required angle is therefore 18°3′38″.

6.8. Angle between two planes when the equations of the planes are given. If the equations of the given planes are in normal form the true angle can be determined as follows:

Example 1. The equations of two planes are

$$\begin{aligned} \tfrac{1}{3}x + \tfrac{2}{3}y + \tfrac{2}{3}z &= 7, \\ \tfrac{2}{7}x + \tfrac{3}{7}y + \tfrac{6}{7}z &= 8. \end{aligned}$$

Find the true angle between the two planes.

The equations are in normal form, because

$$\begin{aligned} (\tfrac{1}{3})^2 + (\tfrac{2}{3})^2 + (\tfrac{2}{3})^2 &= 1, \\ (\tfrac{2}{7})^2 + (\tfrac{3}{7})^2 + (\tfrac{6}{7})^2 &= 1. \end{aligned}$$

Therefore $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{2}{3}$ are the direction cosines of a normal to the first plane, and $\frac{2}{7}$, $\frac{3}{7}$, and $\frac{6}{7}$ are the direction cosines of a normal to the second plane. The angle between the normals to the two given planes is given by

$$\begin{aligned} \cos\theta &= (\tfrac{1}{3})(\tfrac{2}{7}) + (\tfrac{2}{3})(\tfrac{3}{7}) + (\tfrac{2}{3})(\tfrac{6}{7}). \\ \cos\theta &= 0.95238. \\ \theta &= 17°45'13''. \end{aligned}$$

The angle between the normals is equal to the angle between the planes, so the true angle between the two given planes is 17°45′13″.

If the equations of the given planes are in the general form, but not in the normal form, then the true angle between the two planes can be determined as follows:

Example 2. The equations of two planes are

$$\begin{aligned} 3x + 2y - 5z &= 9, \\ 3y + 6z &= 17. \end{aligned}$$

Find the true angle between the two planes.

The direction ratios of a normal to the first plane are $3:2:-5$, and the direction ratios of a normal to the second plane are $0:3:6$. The true angle between the normals is given by

$$\begin{aligned} \cos\theta &= \frac{(3)(0) + (2)(3) + (6)(-5)}{\sqrt{3^2 + 2^2 + (-5)^2}\,\sqrt{0^2 + 3^2 +}} \\ \cos\theta &= -0.58038. \\ &= 125°28'38''. \end{aligned}$$

This is also the true angle between the two planes.

6.9. Direction ratios of the line of intersection of two planes. Consider two planes that intersect. Consider a normal to each of the two planes. Finally, consider a normal to these two

normals. This last normal is parallel to the line of intersection of the two planes (see Fig. 6.7). To find the direction ratios of the line of intersection of two planes, proceed as follows:

1. Find the direction ratios of a normal to one of the given planes.

2. Find the direction ratios of a normal to the other given plane.

3. Find the direction ratios of a normal to these two normals.

The amount of calculation necessary to find the direction ratios

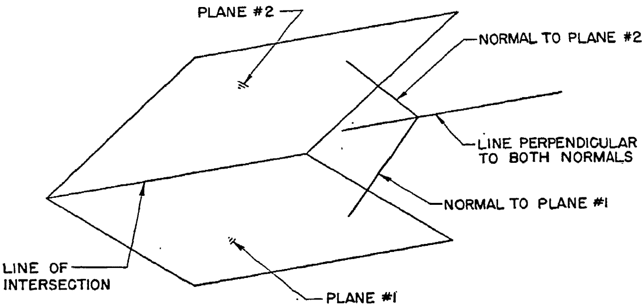


FIG. 6.7.

of the line of intersection of two planes depends upon the ways in which the given planes are determined.

Example 1. The equations of two planes are

$$\begin{aligned} x + 2y + 3z &= 9, \\ 3x + y + 4z &= 10. \end{aligned}$$

Find the direction ratios of the line of intersection of these two planes.

The direction ratios of normals to the given planes are

$$1:2:3,$$
$$3:1:4.$$

To find the direction ratios of a normal to these two normals we cross-multiply, as explained in Art. 5.6.

$$\begin{aligned} (2)(4) - (1)(3) &= 5. \\ (3)(3) - (1)(4) &= 5. \\ (1)(1) - (2)(3) &= -5. \end{aligned}$$

The direction ratios of the line of intersection of the two planes are therefore $5:5:-5$, which can be reduced by dividing by 5 to give $1:1:-1$.

Example 2. A certain wing is rigged by the wing reference plane system. A cant rib in this wing has the equation

$$x + y \tan A = 52.013.$$

Determine the direction ratios of the line of intersection of the given plane with the wing reference plane in rigged position (refer to Fig. 6.8).

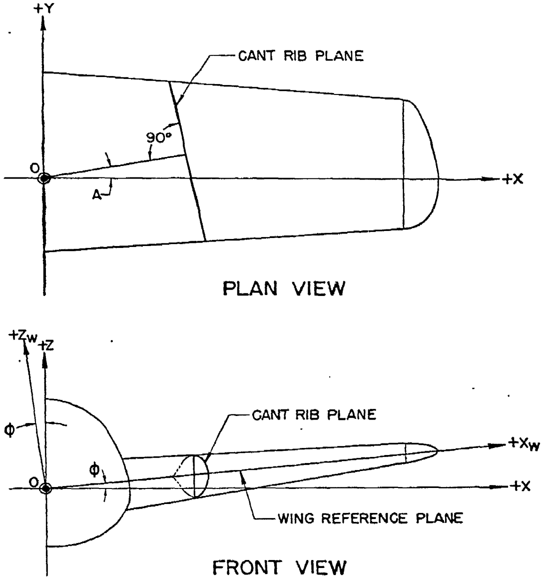


FIG. 6.8.

Since the equation of the cant rib is given, the direction ratios of a normal to the cant rib are the coefficients of x, y, and z:

$$1 : \tan A : 0.$$

The z_w axis is normal to the wing reference plane. The direction ratios of the z_w axis with reference to the rigged axes are

$$-\tan \phi : 0 : 1.$$

The direction ratios of a normal to these two normals may be obtained by cross-multiplying the direction ratios

$$1 : \tan A : 0,$$
$$-\tan \phi : 0 : 1.$$

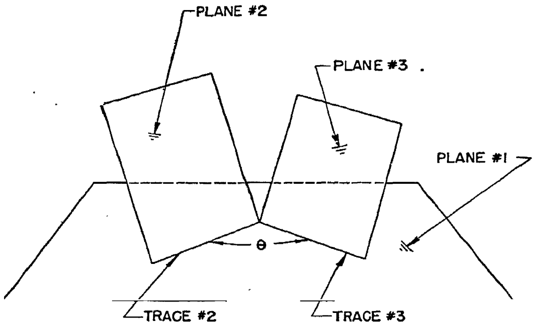
This gives

$$\begin{aligned}(\tan A)(1) - (0)(0) &= \tan A.\\ (0)(-\tan\phi) - (1)(1) &= -1.\\ (1)(0) - (-\tan\phi)(\tan A) &= \tan\phi\tan A.\end{aligned}$$

The direction ratios of the line of intersection of the two planes are then

$$\tan A : -1 : \tan\phi\tan A.$$

The chief use of the direction ratios of the line of intersection of two planes is in the problem of finding the angle made on one plane by the intersections of two other planes. This is ordinarily a rather difficult problem to calculate or to lay out by descriptive



θ = REQUIRED ANGLE
FIG. 6.9.

geometry, but the method described in the next article reduces its solution to a systematic mathematical procedure.

6.10. Angle made on one plane by the intersections of two other planes. To find the angle made on one plane by the traces (intersections) of two other planes, proceed as follows:

1. Find the direction ratios of the line of intersection of the plane with one of the two planes which intersect it.
2. Find the direction ratios of the line of intersection of the plane with the other plane that intersects it.
3. Find the true angle between these two lines of intersection. The procedure will be made clear in the examples (see Fig. 6.9).

Example 1. Find the angle made on the plane $x + 2y + 4z = 6$ by the intersections on it of the two planes $3x - y + 2z = 1$ and $x + 2y - z = 5$.

The direction ratios of a normal to the first plane are 1:2:4, and the direction ratios of a normal to the second plane are 3: −1:2. To find the direction ratios of the line of intersection of these two planes, cross-multiply

$$\begin{matrix} 1: & 2:4 \\ 3: & -1:2. \end{matrix}$$

The results are 8:10: −7. The direction ratios of a normal to the first plane are 1:2:4, and the direction ratios of a normal to the third plane are 1:2: −1.

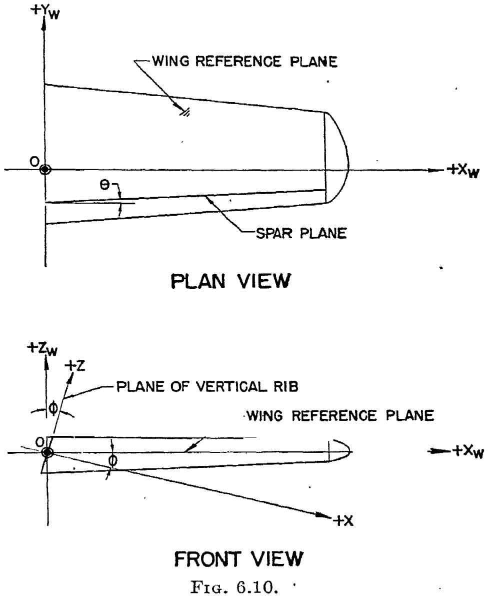


Fig. 6.10.

To find the direction ratios of the line of intersection of the first and third planes, cross-multiply

$$\begin{matrix} 1:2: & 4 \\ 1:2: & -1. \end{matrix}$$

The results are −10:5:0. To summarize, the direction ratios of the two traces (intersections) are 8:10: −7 and −10:5:0. The true angle between these two traces (intersections) is given by

$$\cos\theta = \frac{(8)(-10) + (10)(5) + (-7)(0)}{\sqrt{8^2 + 10^2 + (-7)^2}\sqrt{(-10)^2 + 5^2 + 0^2}}.$$

$$\cos\theta = -0.18386.$$

$$\theta = 100°35'41''.$$

This is the angle made in the first plane by the intersections on it of the other two planes.

Example 2. Find the angle made on a vertical rib plane between the lines of intersection of the wing reference plane and a spar plane, which is perpendicular to the wing reference plane and has a sweepback angle of θ (see Fig. 6.10).

The z_w axis is normal to the wing reference plane. Its direction ratios are

$$0:0:1.$$

The x axis is normal to the plane of the vertical rib. Its direction ratios with reference to the wing reference plane system of axes are

$$1:0:-\tan\phi.$$

The direction ratios of the line of intersection of the wing reference plane and the vertical rib plane are obtained by cross-multiplying

$$\begin{gathered}0:0:1\\ 1:0:-\tan\phi.\end{gathered}$$

The results are $0:1:0$.

The direction ratios of a normal to the vertical rib plane are, as above,

$$1:0:-\tan\phi.$$

The direction ratios of a normal to the spar plane are

$$-\tan\theta:1:0.$$

The direction ratios of the line of intersection of the vertical rib plane and the spar plane are obtained by cross-multiplying

$$\begin{gathered}1:0:-\tan\phi\\ -\tan\theta:1:0.\end{gathered}$$

The results are $\tan\phi:\tan\phi\tan\theta:1$. To summarize, the direction ratios of the two traces (intersections) are

$$\begin{gathered}0:1:0,\\ \tan\phi:\tan\phi\tan\theta:1.\end{gathered}$$

The true angle between these two traces is given by

$$\cos A = \frac{(0)(\tan\phi) + (1)(\tan\phi\tan\theta) + (0)(1)}{\sqrt{0^2+1^2+0^2}\sqrt{\tan^2\phi + \tan^2\phi\tan^2\theta + 1}}.$$

$$\cos A \quad \frac{\tan\phi\tan\theta}{\sqrt{\tan^2\phi + \tan^2\phi\tan^2\theta + 1}}.$$

$$\cos A = \frac{\tan\phi\tan\theta}{\sqrt{\tan^2\phi\tan^2\theta + \sec^2\phi}}$$

This is the answer, but it can be reduced to a simpler form (see Fig. 6.11).

In the triangle in Fig. 6.11, cos A agrees with the result obtained above. Notice that

$$\tan A = \frac{\sec \phi}{\tan \phi \tan \theta}$$

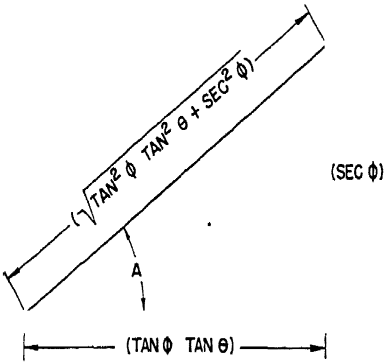


FIG. 6.11.

Since $\sec \phi = \dfrac{1}{\cos \phi}$ and $\tan \phi = \dfrac{\sin \phi}{\cos \phi}$, we have $\dfrac{\sec \phi}{\tan \phi} = \dfrac{1}{\sin \phi}$, and so

$$\tan A = \frac{1}{\sin \phi \tan \theta}$$

This is the final result, which is a formula for the angle made on the plane of a vertical rib by the intersections of the spar plane and the wing reference plane with the plane of the vertical rib. In this formula, ϕ is the angle of dihedral and θ is the sweepback angle of the spar plane.

CHAPTER 7

EQUATIONS OF LINES

In this chapter we shall show how to write the equations of a line. In the case of lines, as well as planes, it is advisable to write the equations of the basic straight lines and to file them for future reference. We shall also consider in this chapter such

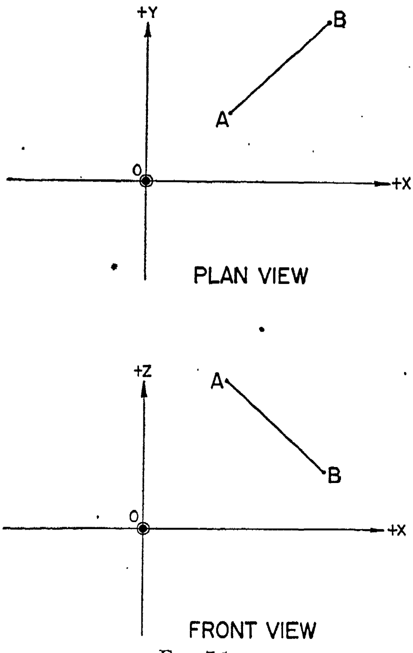


FIG. 7.1.

matters as the distance from a point to a line, the intersection of a line and a plane, the distance between two skew lines in space, and the bisector of an angle in space.

7.1. Equations of lines. A skew line in space is determined by two views of the line, from the standpoint of orthographic projection. Each of these two views represents the projection

of the line on a reference plane. Each of these two views determines an equation, which represents the projection of the line in that view. Therefore the two equations of the two projections will completely determine the line (see Fig. 7.1).

In Fig. 7.1 the line segment AB is shown in two views. These two views of the line segment completely determine the location of the line AB in space. We shall use the terms *line segment* and

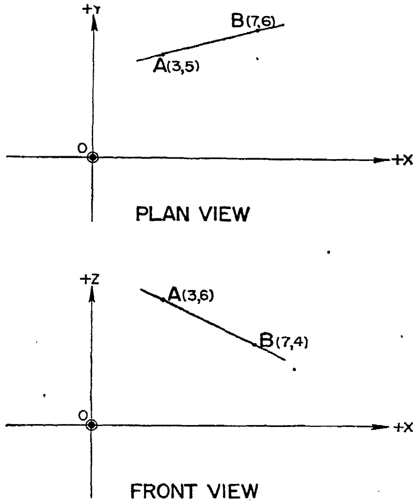


FIG. 7.2.

line interchangeably in this article, since the line segment determines the line. Suppose that the equation of the projection of AB, as shown in the plan view, with reference to the x axis and y axis, is

$$y = 2x - 1,$$

and the equation of the projection of AB, as shown in the front view, with reference to the x axis and z axis, is

$$z = -3x + 8.$$

Then the pair of equations

$$\begin{aligned} y &= 2x - 1 \\ z &= -3x + 8 \end{aligned}$$

completely determines the skew line AB.

Example 1. Write the equations of the line AB in Fig. 7.2.
In the plan view,

$$y - y_1 = m(x - x_1).$$
$$y - 5 = \frac{6 - 5}{7 - 3}(x - 3).$$
$$y = +\tfrac{1}{4}x + 4\tfrac{1}{4}.$$

In the front view,

$$z - z_1 = m(x - x_1).$$
$$z - 6 = \frac{4 - 6}{7 - 3}(x - 3).$$
$$z = -\tfrac{1}{2}x + 7\tfrac{1}{2}.$$

Therefore the equations of AB are

$$y = +\tfrac{1}{4}x + 4\tfrac{1}{4},$$
$$z = -\tfrac{1}{2}x + 7\tfrac{1}{2}.$$

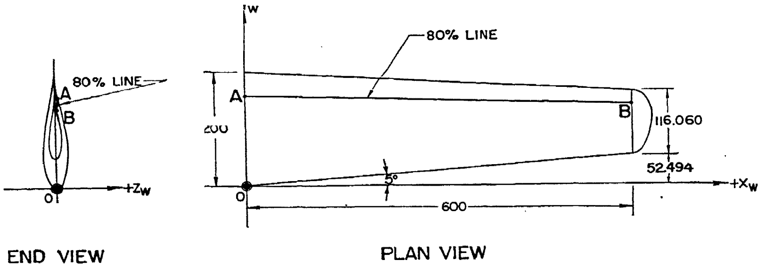


FIG. 7.3.

The equations of a line should be checked. This can be done by substituting the coordinates of the given points in the corresponding equations. If the coordinates satisfy the equation the equation is correct.

Example 2. Find the equations of the 80 per cent line, as determined in Fig. 7.3.

The x_w coordinate of A is 0. The y_w coordinate of A is 80 per cent of 200, or 160. The coordinates of A in the plan view are therefore (0, 160). The x_w coordinate of B is 600. The y_w coordinate of B is

$$52.494 + (0.80)(116.060) = 145.342.$$

The coordinates of B in the plan view are therefore (600, 145.342). Use the point-slope formula for the equation of a straight line.

$$y - y_1 = m(x - x_1).$$
$$y_w - 160 = \frac{145.342 - 160}{600 - 0}(x_w - 0).$$
$$y_w = -0.02443x_w + 160.$$

This is the equation of the projection of AB on the x_wy_w plane.

The z_w coordinate of A in the end (body-plan) view is 9.711, and the y_w coordinate of A in this view is the same as in the plan view, so that the coordinates of A in the end view are (160, 9.711). The z_w coordinate of B in the end view is 5.126 and the y_w coordinate of B in this view is the same as in the plan view, and so the coordinates of B in the end view are (145.342, 5.126). Use the point-slope formula for the equation of a straight line.

$$
\begin{aligned}
y_w - y_1 &= m(z_w - z_1).\\
y_w - 160 &= \frac{160 - 145.342}{9.711 - 5.126}(z_w - 9.711).\\
y_w &= 3.19695z_w + 128.954.
\end{aligned}
$$

This example is typical of the method usually used for writing the equations of a line. The pair of equations

$$
\begin{aligned}
y_w &= -0.02443x_w + 160\\
y_w &= 3.19695z_w + 128.954
\end{aligned}
$$

completely represent the 80 per cent line. If the equation of the 80 per cent line in the front view (x_wz_w plane) is needed, it can be determined as follows: The x_w coordinate of A, as obtained from the plan view, is 0. The z_w coordinate of A, as obtained from the end view, is 9.711. Therefore the coordinates of A in the front view (x_wz_w plane) are (0, 9.711). The x_w coordinate of B in the plan view is 600, and the z_w coordinate of B in the end view is 5.126. The coordinates of B in the front view (x_wz_w plane) are therefore (600, 5.126). Use the point-slope formula for the equation of a line.

$$
\begin{aligned}
- z_1 &= m(x_w - x_1).\\
z_w - 9.711 &= \frac{9.711 - 5.126}{0 - 600}(x_w - 0).\\
z_w &= -0.00764x_w + 9.711.
\end{aligned}
$$

The equations of the 80 per cent line in the three basic orthographic views are therefore

$$
\begin{aligned}
y_w &= -0.02443x_w + 160.\\
y_w &= 3.19695z_w + 128.954.\\
z_w &= -0.00764x_w + 9.711.
\end{aligned}
$$

Any two of these equations would be sufficient to determine the line AB. The equations of per cent lines can be used to develop canted ribs and normal ribs. This procedure will be explained in a subsequent article.

Sometimes a line is determined by two points, as in Examples 1 and 2, and sometimes a line is determined by one point and the projected angles with respect to the reference axes. When this is the case we can proceed as follows:

Example 3. Find the equations of the center line of aileron hinge, as determined in Fig. 7.4.

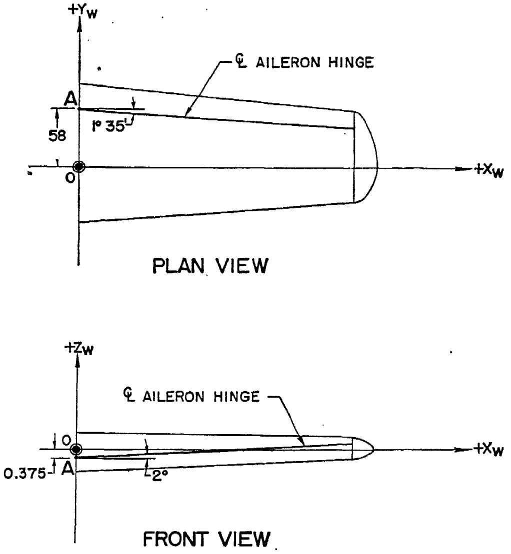


FIG. 7.4.

In the plan view the point A has coordinates (0, 58) and the sweepforward angle is 1°35′, making the slope of the line equal to tan 178°25′ = −0.02764. Use the slope-intercept forms of the equation of a straight line.

$$y = mx + b.$$
$$y_w = -0.02764x_w + 58.$$

In the front view the point A has coordinates (0, −0.375) and the slope of the line is tan 2° = 0.03492. Use the slope-intercept form of the equation of a straight line.

$$z = mx + b.$$
$$z_w = 0.03492x_w - 0.375.$$

The equations of the aileron hinge center line are therefore

$$y_w = -0.02764x_w + 58,$$
$$z_w = 0.03492x_w - 0.375.$$

Let $A_1x + B_1y + C_1z + D_1 = 0$ and

$$A_2x + B_2y + C_2z + D_2 = 0$$

be the equations of two planes that intersect. These equations, when considered simultaneously, represent the line of intersection of the two planes. The equations of a line are therefore of the form

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0.$$

Through any line in space it is possible to pass infinitely many pairs of planes, and there are therefore infinitely many pairs of equations that can represent the same line. Through any line in space it is possible to pass two planes that are each perpendicular to a reference plane. This is done by projecting the line on any two of the three coordinate planes. The equations of these two planes would have only two variables in each equation:

$$A_1x + B_1y = D_1.$$
$$A_2x + C_2z = D_2.$$

If a line is determined by one point and its direction ratios, its equation can be determined as follows: Let the given point be $P_1(x_1, y_1, z_1)$ and let the given direction ratios be $a:b:c$. Then the equations of the line are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}$$

$$\frac{x - x_1}{a} = \frac{z - z_1}{c}$$

$$\frac{y - y_1}{} = \frac{z - z_1}{}$$

These equations can be derived from the following considerations: Let $P(x, y, z)$ be any point on the required line. The direction ratios of PP_1 are therefore

$$x - x_1 : y - y_1 : z - z_1.$$

But the direction ratios of PP_1 are

$$a:b:c.$$

Therefore

$$\frac{x - x_1}{} = \frac{y - y_1}{} = \frac{z - z_1}{}.$$

Example 4. A line is determined by the point (0, 5, 3) and direction ratios $1:-0.2:-0.1$. Write the equations of the line.

Use the equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b},$$
$$\frac{x - x_1}{a} = \frac{z - z_1}{c},$$
$$\frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Substituting in these equations, we obtain

$$\frac{x - 0}{1} = \frac{y - 5}{-0.2},$$
$$\frac{x - 0}{1} = \frac{z - 3}{-0.1},$$
$$\frac{y - 5}{-0.2} = \frac{z - 3}{-0.1}.$$

These equations can be simplified to give

$$\begin{aligned} y &= -0.2x + 5, \\ z &= -0.1x + 3, \\ z &= 0.5y + 0.5. \end{aligned}$$

These equations represent the projections of the line on the xy plane, xz plane, and yz plane, respectively. Any two of them would be sufficient to determine the line.

Example 5. The front spar top lofted line passes through the point (0, 15, 7), and its direction ratios are $1:-0.14933:0.08660$. Write its equations.

Use the equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b},$$
$$\frac{x - x_1}{a} = \frac{z - z_1}{c},$$
$$\frac{y - y_1}{} = \frac{z - z_1}{}.$$

Substituting in these equations, we obtain

$$\frac{x_w - 0}{1} = \frac{y_w - 15}{-0.14933},$$
$$\frac{x_w - 0}{1} = \frac{z_w - 7}{0.08660},$$
$$\frac{y_w - 15}{-0.14933} = \frac{z_w - 7}{0.08660}.$$

These equations can be simplified to give

$$\begin{aligned} y_w &= -0.14933x_w + 15, \\ z_w &= 0.08660x_w + 7. \end{aligned}$$

These equations represent the projections of the front spar top lofted line on the x_wy_w plane and x_wz_w plane, respectively. These two equations are sufficient to determine the line.

If a line is determined by two points the method illustrated above in Examples 4 and 5 can be used to write the equations of the line.

Example 6. A spar lofted line is determined by the two points $A(-1, 63, 6)$ and $(239, 15, 30)$. Find the equations of this line.

The direction ratios of the required line are

$$239 - (-1):15 - 63:30 - 6 \qquad \text{or} \qquad 1:-0.2:0.1.$$

Use the equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}$$

$$\frac{x - x_1}{a} = \frac{z - z_1}{c},$$

$$\frac{y - y_1}{} = \frac{z - z_1}{}.$$

Either of the two given points can be used as (x_1, y_1, z_1). The results will be the same no matter which one of the two points is chosen to be (x_1, y_1, z_1).

$$\frac{x_w + 1}{1} = \frac{y_w - 63}{-0.2}.$$

$$\frac{x_w + 1}{} \quad \frac{z_w - 6}{0.1},$$

$$\frac{y_w + 63}{-0.2} \quad \frac{z_w - 6}{0.1}$$

These equations can be simplified to give

$$\begin{aligned} y_w &= -0.2x_w + 62.8, \\ z_w &= 0.1x_w + 6.1. \end{aligned}$$

These equations represent the projections of the line on the x_wy_w plane and x_wz_w plane, respectively. These two equations are sufficient to determine the line.

7.2. Special cases of equations of lines. If a line is level in one view, then it is in the plane of the paper in the other view, and its equations take special forms.

Example. From the information in Fig. 7.5 find the equations of the line AB.

The coordinates of A are (0, 8, 6). The direction ratios of AB can be obtained by assuming that AB in the front view is 1 unit long. Then the

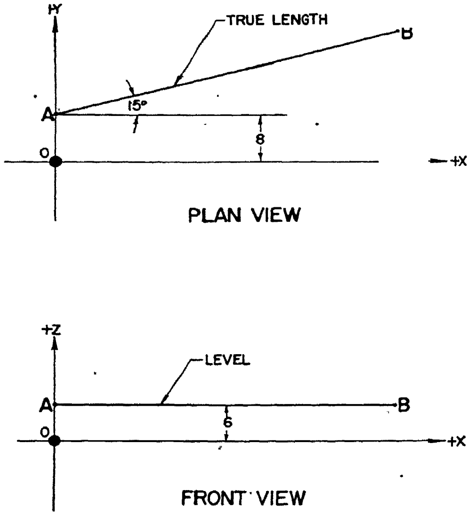


FIG. 7.5.

coordinates of B are 1, 8 + tan 15°, 6. The coordinates of B are therefore (1, 8.26795, 6). The direction ratios of AB are

$$1 - 0{:}8.26795 - 8{:}6 - 6 \qquad \text{or} \qquad 1{:}0.26795{:}0.$$

The third direction ratio is 0 because the line's projection is level in the xz plane. The equations of AB are

$$\frac{x-0}{1} = \frac{y-8}{0.26795}$$

$$\frac{x-0}{1} = \frac{z-6}{0},$$

$$\frac{y-.8}{0.26795} = \frac{z-6}{0}.$$

Now $\frac{z-6}{0}$ is undefined because division by zero is excluded. By examining the two given views we find that the equation of AB in the front view is $z = 6$, since the line's projection is 6 units above the x axis. Likewise the equation of the projection in the yz plane would be level, and its equation

would also be $z = 6$. The equation of the line AB in the plan view is

$$y = 0.26795x + 8.$$

That is, the equations of the line AB are

$$\begin{aligned} y &= 0.26795x + 8, \\ z &= 6. \end{aligned}$$

In general, the equations of a line whose projection is level in the view showing the xz plane and whose projection is not level in the view showing the xy plane is of the form

$$\begin{aligned} y &= mx + b, \\ z &= k, \end{aligned}$$

where m, b, and k are constants. This example is typical of other cases in which the projection of a line is level in one view and not level in another view.

The equations of a line parallel to the x axis are of the form

$$\begin{aligned} y &= D_1, \\ z &= D_2. \end{aligned}$$

The equations of a line parallel to the y axis are of the form

$$\begin{aligned} x &= D_1, \\ z &= D_2. \end{aligned}$$

The equations of a line parallel to the z axis are of the form

$$\begin{aligned} x &= D_1, \\ y &= D_2. \end{aligned}$$

In these equations D_1 and D_2 represent constants and may have any numerical values.

7.3. Distance from a point to a line. In Fig. 7.6, the given point is $P_1(x_1, y_1, z_1)$ and the given line is determined by the two points $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$. The problem is to find the distance from P_1 to the line through P_2P_3. By the "distance" we mean the perpendicular distance, and so P_1A is perpendicular to the line P_2P_3. The distance P_1A is also the shortest distance from the point P_1 to the line P_2P_3 (see Fig. 7.6). The point P_1 can be any point in space, and the line P_2P_3 can be any skew line in space. This is more general than the case described in Art. 2.9.

The procedure is as follows:

1. Select *any point* on P_2P_3, say P_2.
2. Calculate the cosine of the true angle between P_1P_2 and P_2P_3. Denote it by $\cos \theta$.

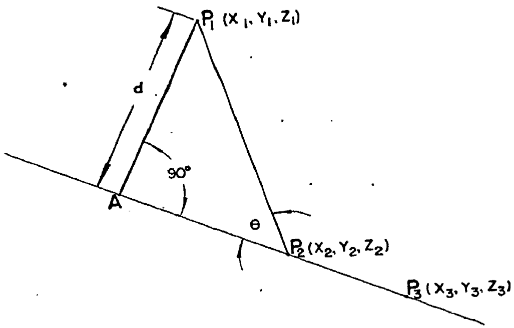


FIG. 7.6.

3. Calculate the true length P_1P_2.
4. Determine P_1A from the formula

$$d = P_1A = P_1P_2 \sin \theta = P_1P_2 \sqrt{1 - \cos^2 \theta}.$$

This formula is obvious from Fig. 7.6, since

$$\frac{d}{P_1P_2} = \sin \theta.$$
$$d = P_1P_2 \sin \theta.$$

Notice that P_2 can be any point on the given line.

Example 1. Find the distance from the point $A(10, 5, 13)$ to the line $B(10, 2, 9)$ $C(4, 0, 6)$. See Fig. 7.7.

The direction cosines of CB are

$$\tfrac{6}{7}, \tfrac{2}{7}, \tfrac{3}{7}.$$

The direction cosines of BA are

$$0, \tfrac{3}{5}, \tfrac{4}{5}.$$

The true angle between AB and BC is given by

$$\cos \theta = (\tfrac{6}{7})(0) + (\tfrac{2}{7})(\tfrac{3}{5}) + (\tfrac{3}{7})(\tfrac{4}{5}).$$
$$\cos \theta = \tfrac{18}{35}.$$

The true length of AB is 5.
The distance AD is given by

$$d = AB\sqrt{1 - \cos^2\theta}.$$
$$d = 5\sqrt{1 - (\tfrac{18}{35})^2}.$$
$$d = 4.288.$$

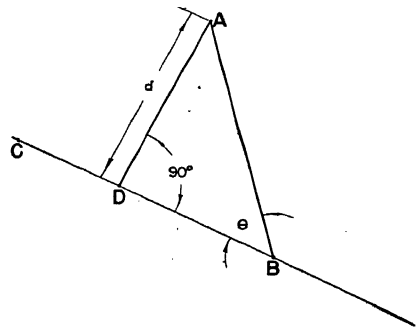


FIG. 7.7.

Example 2. Find the distance from the point P to the line AB in Fig. 7.8. The direction ratios of AB are

$$-\tan 12°:1:-\tan 8°, \qquad \text{or} \qquad -0.21256:1:-0.14054.$$

The direction cosines of AB are

$$-0.20598,\ 0.96904,\ -0\ 13619.$$

The direction ratios of AP are

$$-9-(-15):25-10:5-12, \qquad \text{or} \qquad 6:15:-7.$$

The direction cosines of AP are

$$0.34078,\ 0.85194,\ -0.39757.$$

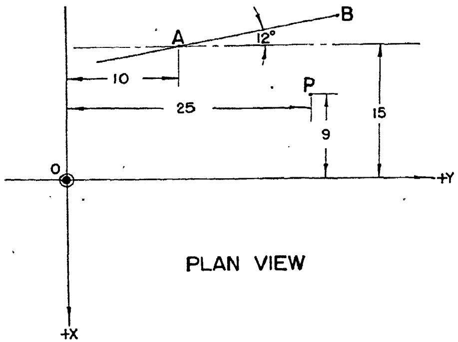
The true angle between AP and AB is given by

$$\cos\theta = (-0.20598)(0.34078)+(0.96904)(0.85194)+(-0.13619)(-0.39757).$$
$$\cos\theta = 0.80952.$$

The true length AP is 17.607. Notice that this true length was obtained in the process of calculating the direction cosines of AP.

The required distance is given by

$$d = AP\sqrt{1 - \cos^2\theta}.$$
$$d = (17.607)(0.58709).$$
$$d = 10.337.$$



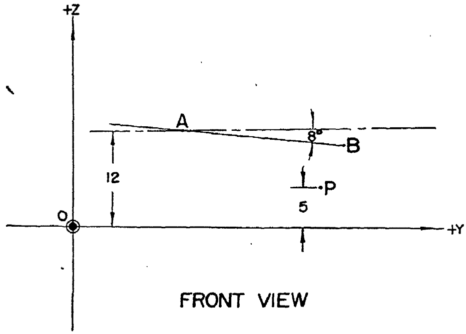


FIG. 7.8.

Example 3. Find the amount of clearance (shortest distance) between the point $P(30, 295, -3)$ and the control cable determined by the two points $A(20, -30, 10)$, $B(35, 300, -4)$.

The direction cosines of BA are

$$-0.04536, -0.99807, 0.04234.$$

The direction cosines of PB are

$$-0.70014, -0.70014, 0.14003.$$

The true angle between PB and AB is given by

$$\cos\theta = (-0.04536)(-0.70014) + (-0.99807)(-0.70014) + (0.04234)(0.14003) = 0.73648.$$

The true length of PB is 7.141.
The clearance is given by

$$d = PB\sqrt{1-\cos^2\theta}.$$
$$d = (7.141)(0.67646).$$
$$d = 4.831.$$

This method for finding the true distance from a point to a line is general and finds many applications on the airplane. Notice particularly that in Fig. 7.6 the direction cosines of the line AP_2 are not sufficient to determine the line AP_2. In addition, one point on the line AP_2 must be known in order to fix the exact location of the line in space. In certain other problems, such as determining the true angle between two lines in space, the direction cosines are sufficient, because in such problems we are calculating the angle between two directions in space, and it is necessary to know the directions of the lines, but not their exact locations in space.

7.4. True (shortest) distance between two lines. Consider the line AB, whose direction cosines are $\cos\alpha_1$, $\cos\beta_1$, and $\cos\gamma_1$, and the line CD, whose direction cosines are $\cos\alpha_2$, $\cos\beta_2$, and $\cos\gamma_2$ (see Fig. 7.9). Calculate the direction cosines of a line perpendicular to AB and CD by cross-multiplying

$$\cos\alpha_1 : \cos\beta_1 : \cos\gamma_1$$
$$\cos\alpha_2 : \cos\beta_2 : \cos\gamma_2$$

and reducing the resulting direction ratios to direction cosines. Let us denote these direction cosines by $\cos\alpha$, $\cos\beta$, and $\cos\gamma$.

Consider the plane that contains the line AB and is normal to the common perpendicular between the two given lines. The direction cosines of a normal to this plane are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. Let $P_1(x_1, y_1, z_1)$ be any point on the line AB. The perpendicular distance from the origin to the plane is $x_1\cos\alpha + y_1\cos\beta + z_1\cos\gamma$. The equation of the plane is therefore

$$x\cos\alpha + y\cos\beta + z\cos\gamma = x_1\cos\alpha + y_1\cos\beta + z_1\cos\gamma.$$

This equation is in the normal form, since the coefficients of x, y, and z are the direction cosines of a normal to the plane, and

the expression on the right-hand side of the equation is equal to the distance from the origin to the plane.

Consider the plane that contains the line CD and is normal to the common perpendicular between the two given lines. Its equation is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = x_2 \cos \alpha + y_2 \cos \beta + z_2 \cos \gamma,$$

where $P_2(x_2, y_2, z_2)$ is any point on the line CD. This equation is in the normal form.

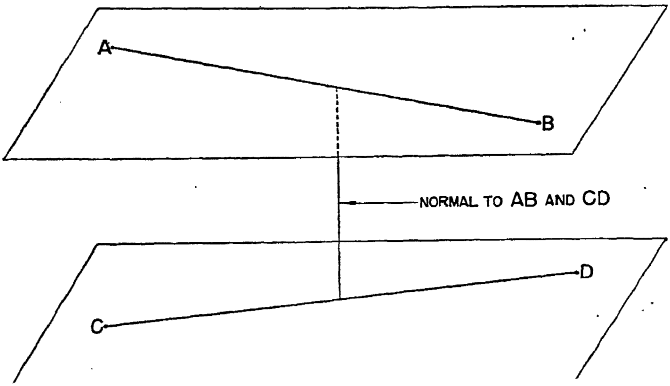


FIG. 7.9.

The distance between the two planes is equal to the length of the common perpendicular. Notice that the two planes are parallel, since they are both perpendicular to the same line.

The distance between AB and CD is therefore

$$\begin{aligned} d &= (x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma) - (x_2 \cos \alpha + y_2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \cos \beta + z_2 \cos \gamma) \\ &= (x_1 - x_2) \cos \alpha + (y_1 - y_2) \cos \beta + (z_1 - z_2) \cos \gamma. \end{aligned}$$

Example 1. Find the shortest distance (length of the common perpendicular) between the two lines $A(10, 8, 6)$ $B(10, 2, -2)$ and $C(20, 15, 5)$ $D(16, 9, -7)$.

The direction ratios of BA are

$$10 - 10:8 - 2:6 - (-2) \qquad 0:6:8 \qquad \text{or} \qquad 0:3:4.$$

The direction ratios of DC are

$$20 - 16:15 - 9:5 - (-7) \qquad \text{or} \qquad 4:6:12 \qquad \text{or} \qquad 2:3:6.$$

The direction ratios of a normal to BA and DC are obtained by cross-multiplying

$$0:3:4$$
$$2:3:6.$$

The results are $6:8:-6$. These can be reduced to direction cosines to give

$$0.51450:0.68599:-0.51450.$$

Now A is a point on AB, and C is a point on CD. The shortest distance between AB and CD is therefore given by

$$d = (x_1 - x_2) \cos \alpha + (y_1 - y_2) \cos \beta + (z_1 - z_2) \cos \gamma.$$
$$d = (0.51450)(20 - 10) + (0.68599)(15 - 8) + (-0.51450)(5 - 6).$$
$$d = (0.51450)(10) + (0.68599)(7) + (-0.51450)(-1).$$
$$d = 10.461.$$

To find the distance between two lines in space, proceed as follows:

1. Determine the direction ratios of each of the two given lines.
2. Find the *direction cosines* of a line normal to both of the given lines. Denote these direction cosines by $\cos \alpha$, $\cos \beta$, $\cos \gamma$.
3. Select any point on one of the given lines, say (x_1, y_1, z_1). Select any point on the other given line, say (x_2, y_2, z_2).
4. The distance is given by

$$d = (x_1 - x_2) \cos \alpha + (y_1 - y_2) \cos \beta + (z_1 - z_2) \cos \gamma.$$

Example 2. Two control cables are determined as shown in Fig. 7.10. Calculate the amount of clearance, if any, between the control cables.

Set up an auxiliary system of axes, as shown. It is helpful to take one of the given points as the origin, because then its coordinates will be (0, 0, 0) and the zeros will help to simplify the calculations.

The direction ratios of AB are

$$-\tan 20°:1:-\tan 15° \qquad -0.36397:1:-0.26795.$$

The direction ratios of CD are

$$\tan 35°:1:\tan 38° \qquad \text{or} \qquad 0.70021:1:0.78129.$$

The direction ratios of a normal to AB and CD are obtained by cross-multiplying

$$-0.36397:1:-0.26795$$
$$0.70021:1:0.78129.$$

The direction ratios of the normal are

$$1.04924:0.09674:-1.06418,$$

and its direction cosines are

$$0.70062, 0.06460, -0.71060.$$

A point on AB is $A(0, 0, 0)$ and a point on CD is $C(-8, 4, -5)$.

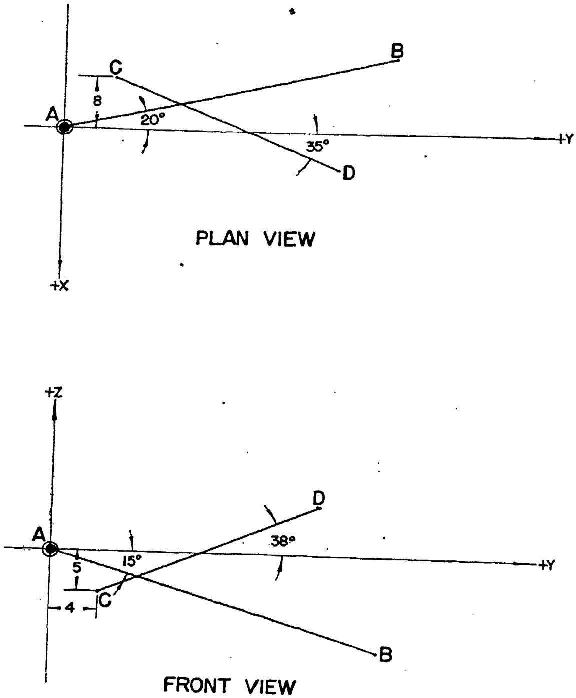


FIG. 7.10.

The required distance is given by

$$d = (x_1 - x_2) \cos \alpha + (y_1 - y_2) \cos \beta + (z_1 - z_2) \cos \gamma.$$
$$d = (0.70062)(-8) + (0.06460)(4) + (-0.71060)(-5).$$
$$d = -1.794.$$

The clearance between the control cables is 1.794 in.

7.5. Intersection of a line and a plane. It is often necessary to determine the point where a given line intersects a given plane.

The equation of a plane in the general form is

$$Ax + By + Cz + D = 0.$$

The equations of a line may be written in the form

$$\begin{aligned} Ex + Fy &= G, \\ Hx + Jz &= K. \end{aligned}$$

To find the point of intersection of the line and the plane, solve these three equations simultaneously for x, y, and z.

Example 1. Find the point where the line

$$\begin{aligned} x + 2y &= 3 \\ 4x + 5z &= 8 \end{aligned}$$

intersects the plane

$$x - 2y + z = 7.$$

Eliminate z from the second and third equations by multiplying the third equation by 5 and subtracting the result from the second equation:

$$\begin{aligned} 4x + 5z &= 8. \\ 5x \quad 10y + 5z &= 35. \\ -x + 10y &= -27. \end{aligned}$$

Solve this equation simultaneously with the first equation.

$$\begin{aligned} x + 2y &= 3. \\ -x + 10y &= -27. \\ \hline 12y &= -24. \\ y &= -2. \end{aligned}$$

Substitute this value for y in the first given equation and solve for x:

$$\begin{aligned} x - 4 &= 3. \\ x &= 7. \end{aligned}$$

Substitute this value for x in the second given equation and solve for z:

$$\begin{aligned} 28 + 5z &= 8. \\ 5z &= -20. \\ z &= -4. \end{aligned}$$

The required point of intersection is $(7, -2, -4)$. These values of x, y, z check in each of the three given equations.

Example 2. The equations of the 80 per cent top lofted line are

$$\begin{aligned} y_w &= -0.02443x_w + 160, \\ z_w &= -0.00764x_w + 9.711. \end{aligned}$$

The equation of the plane of a skew rib in the wing is

$$x_w - 0.03491y_w + 0.12187z_w = 97.459.$$

Find the point where the 80 per cent top lofted line pierces the plane of the rib. Substituting the equations of the line in the equation of the plane of the skew rib,

$$x_w - 0.03491(-0.02443x_w + 160) + (0.12187)(-0.00764x_w + 9.711) = 97.459.$$

$$x_w = 101.869.$$

Substitute $x_w = 101.869$ in the equations of the 80 per cent top lofted line. Then $y_w = 157.511$ and $z_w = 8.933$. The coordinates of the point of intersection of the 80 per cent top lofted line with the double-skewed rib are therefore

$$\begin{aligned} x_w &= 101.869, \\ y_w &= 157.511, \\ z_w &= 8.933. \end{aligned}$$

The intersection of a line and a plane is often simpler than it is in Examples 1 and 2.

Example 3. Find the coordinates of the point of intersection of the 15 per cent top lofted line, whose equations are

$$\begin{aligned} y_w &= 0.03750x_w - 30.000, \\ z_w &= -0.02140x_w + 13.342, \end{aligned}$$

and the normal rib plane $x_w = 10.000$.

In this case, merely substitute $x_w = 10.000$ in the equations of the line. The results are

$$\begin{aligned} y_w &= -29.625, \\ z_w &= 13.128. \end{aligned}$$

The coordinates of the point of intersection of the line and plane are (10.000, −29.625, 13.128).

7.6. Direction ratios of a line when its equations are given. If the equations of a line are

$$\begin{aligned} y &= mx + b \\ z &= nx + c \end{aligned}$$

the direction ratios of the line are

$$1:m:n.$$

If the equations of a line are

$$\begin{aligned} x &= my + b \\ z &= ny + c \end{aligned}$$

the direction ratios of the line are

$$m:1:n.$$

If the equations of a line are

$$x = mz + b$$
$$y = nz + c$$

the direction ratios of the line are

$$m:n:1.$$

Notice that in each case the one is in the position of the letter common to both equations. For example, in the first set of equations, at the beginning of this article, the x is the variable present in both of the given equations, and the one in the set of the direction ratios is therefore in the position of the x, *i.e.*, the first ratio. Also, notice that m and n occupy the positions of the variables on the left sides of the given equations. For example, in the first set of equations, the m is in the equation with the variable y on the left side of the equation, and so it occupies the y (second) position in the set of direction ratios.

Example 1. Find the direction ratios of the line whose equations are

$$y = -2x + 3,$$
$$z = 4x + 5.$$

The result is

$$1:-2:4.$$

Example 2. Find the direction ratios of the line whose equations are

$$x = 3y - 5,$$
$$z = -2y + 7.$$

The result is

$$3:1:-2.$$

Example 3. Find the direction ratios of the line whose equations are

$$x = -4z + 2,$$
$$y = 2z + 3.$$

The result is

$$-4:2:1.$$

Example 4. The equations of a certain per cent line in a certain wing are

$$y_w = -0.14986x_w + 67.125,$$
$$z_w = 0.12875y_w + 45.839.$$

Find the direction ratios of this per cent line.

The y_w is the variable contained in both of the given equations. However, the first equation is expressed in such a way that y_w is given in terms of x_w. The direction ratios are

$$\frac{1}{-0.14986}:1:0.12875.$$

7.7. Bisector of the angle between two lines. Consider the problem of finding the direction ratios of the bisector of the angle between two intersecting lines in space (see Fig. 7.11).

In Fig. 7.11 the two given intersecting lines in space are AB and AC. The required bisector is AD. Suppose that the direction cosines of AB are a, b, c and the direction cosines of AC are d, e, f. The direction ratios of the bisector of angle BAC are

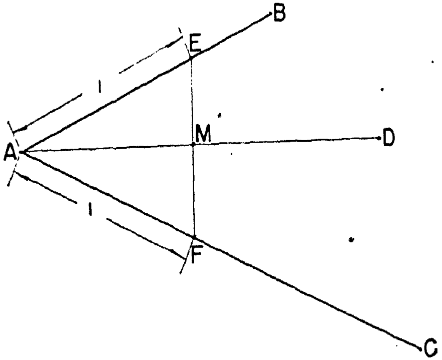


FIG. 7.11.

$\frac{a+d}{2} : \frac{b+e}{2} : \frac{c+f}{2}$. To prove this, assume a set of axes with A as the origin. Then the direction cosines of AB become the coordinates of a point, say E, on AB one unit distant from A. Likewise, the direction cosines of AC become the coordinates of a point, say F, on AC one unit distant from A. The point M, which lies on the bisector, is the mid-point of EF, and so its coordinates are $\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}$. Since A has the coordinates (0, 0, 0), then the direction ratios of AM are $\frac{a+d}{2} : \frac{b+e}{2} : \frac{c+f}{2}$.

CHAPTER 8

TRANSLATION AND ROTATION OF AXES

As pointed out in Chap. 3, several different sets of axes are used on the airplane. These sets of axes are related to each other in certain ways. For example, it is usually possible to move the rigged axes to the vertical stabilizer axes by a motion involving translation only. Also, it is possible to change the rigged axes to the wing axes of the wing reference plane system by a motion involving a rotation through one angle only, namely, the angle of dihedral. In the case of a wing lofted by the wing chord plane system, it is possible to change the rigged axes to the wing chord plane axes by two successive rotations, namely, a rotation through the angle of incidence followed by a rotation through the angle of dihedral. Sometimes, as in the case of the rigged axes and nacelle axes, it is necessary to translate the rigged axes to the nacelle origin and then rotate the axes to the nacelle position. It is the purpose of this chapter to study mathematically these relations among the various sets of axes.

If the coordinates of a point are known with respect to a certain system of axes it is sometimes necessary to be able to calculate its coordinates with respect to a different set of axes, which may be related to the original set of axes by translation, rotation, or a combination of both. This conversion of coordinates from one system of axes to another is accomplished by formulas that will be derived and explained in this chapter. These formulas are necessary in many situations. For example, suppose that a certain line is determined with respect to the rigged system of axes and another line is determined in the wing reference plane system of axes, and that it is required to find the true angle between these two lines. Before the methods developed in Chap. 4 for finding the true angle between two lines can be used, the coordinates of the points determining the two lines must be converted to the same system of axes. Similar conversions must be made for any problem in which part of the information is

given with relation to one set of axes and part with relation to a different set of axes.

The simplicity of the formulas for translating and rotating axes and the ease with which coordinates are converted from one system to another are among the most valuable features of analytic geometry as applied to the airplane.

8.1. Translation of axes in a plane. Consider the two sets of axes shown in Fig. 8.1. The new set of axes x', y' is obtained

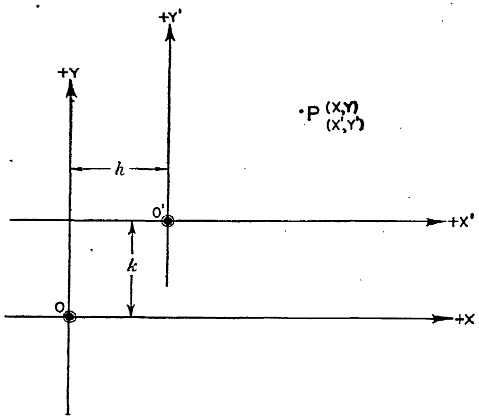


FIG. 8.1.

from the original set of axes x, y by a motion of translation. The positive directions of the x axis and y axis are parallel to the positive directions of the x' axis and y' axis, respectively. Let the coordinates of the new origin O' be (h, k) with respect to the original set of axes. Consider the point P, with coordinates (x, y) with respect to the original set of axes and coordinates (x', y') with respect to the new set of axes. From Fig. 8.1,

$$x - x' = h,$$
$$y - y' = k.$$

These equations can be solved for x', y' to give

$$x' = x - h,$$
$$y' = y - k.$$

The equations can be solved for x, y to give

$$x = x' + h,$$
$$y = y' + k.$$

The preceding two sets of equations enable us to find the coordinates of P with respect to the x', y' set of axes when its coordinates with respect to the x, y set are given, and vice versa.

Example 1. The origin of an x', y' set of axes has coordinates (8, 6) with respect to an x, y set of axes. A certain point P has coordinates (15, 23)

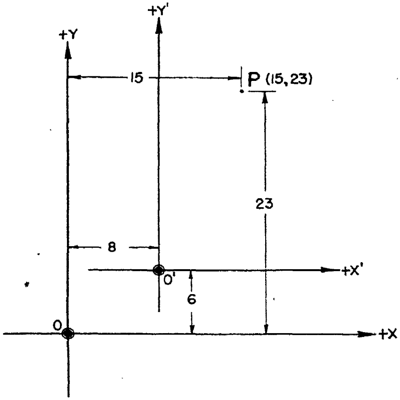


FIG. 8.2.

with respect to the x, y set of axes. Find the coordinates of P with respect to the x', y' set of axes (see Fig. 8.2).

$$\begin{aligned} h &= 8, \\ k &= 6. \\ x &= 15, \\ y &= 23. \\ x' &= x - h, \\ y' &= y - k. \\ x' &= 15 - 8 = 7, \\ y' &= 23 - 6 = 17. \end{aligned}$$

The coordinates of P with respect to the x', y' set of axes are therefore (7, 17).

Example 2. In Fig. 8.3, the point P has coordinates (15, 10) with respect to the x, z system of axes shown. The origin of the x', z' system of axes

has coordinates $(-25, 40)$, with respect to the x, z system of axes. Find the coordinates of the point P with respect to the x', z' system of axes.

$$\begin{aligned} h &= -25, \\ k &= 40. \\ x' &= x + 25, \\ z' &= z - 40. \\ x' &= 15 + 25 = 40, \\ z' &= 10 - 40 = -30. \end{aligned}$$

The coordinates of the point P with respect to the x', z' set of axes are therefore $(40, -30)$.

Example 3. In Fig. 8.3, the fuselage contour is a circle with center at the origin of the x', z' set of axes and with a radius of 50. The equation of

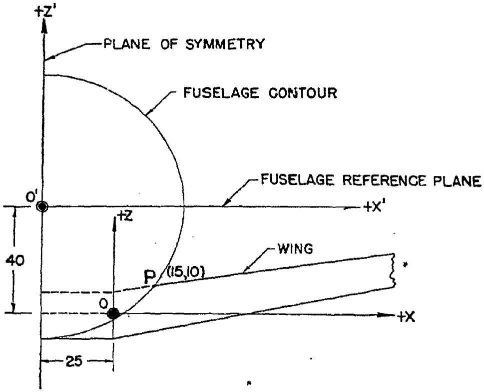


FIG. 8.3.

the fuselage contour with respect to the x', z' set of axes is therefore

$$x'^2 + z'^2 = 2{,}500.$$

Find the equation of the fuselage contour with respect to the x, z set of axes.

$$\begin{aligned} x' &= x + 25, \\ z' &= z - 40. \end{aligned}$$

Substitute these values for x', z' in the equation of the circle.

$$\begin{aligned} (x + 25)^2 + (z - 40)^2 &= 2{,}500. \\ x^2 + 50x + 625 + z^2 - 80z + 1{,}600 &= 2{,}500. \\ x^2 + z^2 + 50x - 80z &= 275. \end{aligned}$$

This is the equation of the fuselage contour with respect to the x, z set of axes.

If the equation

$$x^2 + z^2 + 50x - 80z = 275$$

were given, representing the circle with respect to the x, z axes, then the equation could be simplified by the substitutions

$$x = x' - 25,$$
$$z = z' + 40.$$

The simplified equation would be

$$x'^2 + z'^2 = 2{,}500,$$

which would represent the circle with respect to the x', z' axes.

8.2. Translation of axes in space. Consider a set of axes and a point P whose coordinates are (x, y, z) with respect to this set

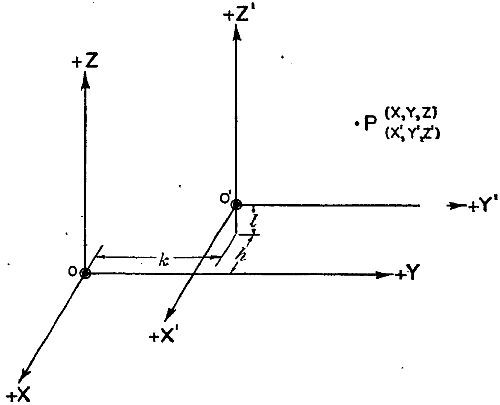


FIG. 8.4.

of axes. Consider a new set of axes having the same positive directions as the original set of axes but having a different origin. The new set of axes can be obtained from the original set of axes by motions of translation alone. Suppose that the coordinates of the origin of the new set of axes are (h, k, l) with respect to the original set of axes. Suppose that the coordinates of P with respect to the new set of axes are (x', y', z'). See Fig. 8.4. Then we have, from Fig. 8.4,

$$x - x' = h,$$
$$y - y' = k,$$
$$z - z' = l.$$

These equations can be solved for x, y, z:

$$x = x' + h,$$
$$y = y' + k,$$
$$z = z' + l.$$

They can also be solved for x', y', z':

$$x' = x - h,$$
$$y' = y - k,$$
$$z' = z - l.$$

The last two sets of equations can be used to convert the coordinates of a point from the new set of axes to the original set of axes, and from the original set of axes to the new set of axes, respectively.

Example 1. The origin of the new system of axes has coordinates (3, 4, 6) with respect to the original system of axes. Find the coordinates of a point $P(35, 28, 16)$ with respect to the new set of axes.

$$x' = x - h,$$
$$y' = y - k,$$
$$z' = z - l.$$
$$h = 3, \qquad k = 4, \qquad l = 6.$$
$$x = 35, \qquad y = 28, \qquad z = 16.$$
$$x' = 35 - 3 = 32.$$
$$y' = 28 - 4 = 24.$$
$$z' = 16 - 6 = 10.$$

The coordinates of the point with respect to the new (translated) system of axes are (32, 24, 10).

Example 2. The equation of a plane with respect to the original set of axes in Example 1 is

$$3x - 2y - 7z = 5.$$

Find the equation of this plane with respect to the new set of axes.

$$x = x' + 3,$$
$$y = y' + 4,$$
$$z = z' + 6.$$

Substitute $x' + 3$ for x, $y' + 4$ for y, and $z' + 6$ for z in the equation of the plane.

$$3(x' + 3) - 2(y' + 4) - 7(z' + 6) = 5.$$

This result can be simplified by multiplying and collecting terms to give

$$3x' - 2y' - 7z' = 46.$$

This is the equation of the given plane with respect to the new set of axes.

Example 3. Consider the rigged system of axes and the system of axes for the vertical stabilizer and rudder shown in Fig. 8.5. See also Fig. 3.14.

The coordinates of a certain point in the vertical stabilizer and rudder system of axes are (12, 4, 23). Find the coordinates of this point with respect to

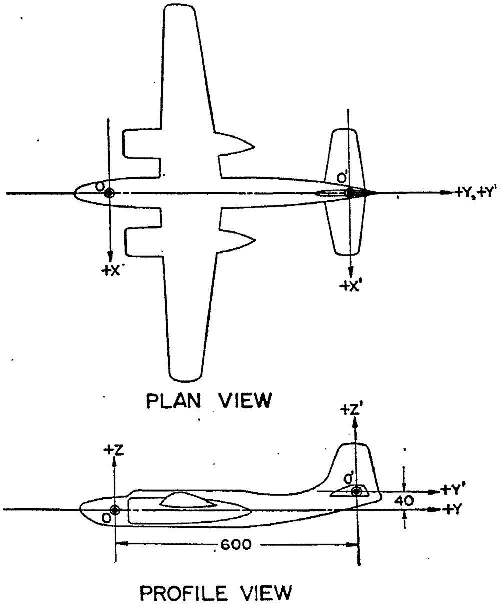


FIG. 8.5.

the rigged system of axes.

$$x = x',$$
$$y = y' + 600,$$
$$z = z' + 40.$$
$$x = 12,$$
$$y = 4 + 600 = 604,$$
$$z = 23 + 40 = 63.$$

Example 4. In Fig. 8.5 the coordinates of a certain point with respect to the rigged system of axes are (8, 612, 44). Find the coordinates of this point with respect to the vertical stabilizer and rudder system of axes.

$$x' = x,$$
$$y' = y - 600,$$
$$z' = z - 40.$$

$$x' = 8,$$
$$y' = 612 - 600 = 12,$$
$$z' = 44 - 40 = 4.$$

The coordinates of the point with respect to the x', y', z' system of axes are (8, 12, 4).

8.3. Rotation of axes (wing reference plane system). The relation between the rigged axes and the wing reference plane axes is given by the following table of direction cosines of the axes with respect to each other:

	x_w	y_w	z_w
x	$\cos \phi$	0	$-\sin \phi$
y	0	1	0
z	$\sin \phi$	0	$\cos \phi$

This table was derived and discussed in Art. 5.2. The relation between the rigged axes and wing reference plane axes is shown in Fig. 8.6. See also Fig. 3.12.

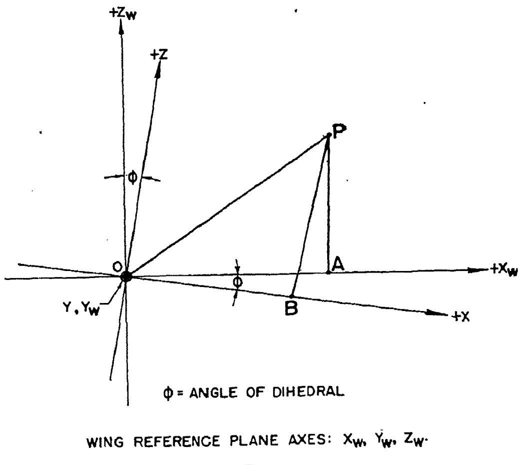


FIG. 8.6.

In Fig. 8.6, the projection of OP on the x_w axis is equal to the sum of the projections of OB on the x_w axis and BP on the x_w axis, since OP is the closing-line segment of the broken line OBP.

The projection of OB on the x_w axis is equal to $OB \cos \phi$, or $x \cos \phi$, since $OB = x$. The projection of BP on the x_w axis is $BP \cos (90° - \phi)$, or $BP \sin \phi$, or $z \sin \phi$, since $BP = z$. The projection of OP on the x_w axis is OA, or x_w, since $OA = x_w$. Therefore

$$x_w = x \cos \phi + z \sin \phi.$$

Notice that y does not occur in this result, because OA, OB, BP, and OP all lie in the plane of the paper and the y axis is perpendicular to the plane of the paper.

Since the y axis and y_w axis coincide, we have

$$y_w = y.$$

The projection of OP on the z_w axis is equal to the sum of the projections of OB on the z_w axis and BP on the z_w axis, since OP is the closing-line segment of the broken line OBP. The projection of OB on the z_w axis is $OB \cos (90° + \phi)$, or $-OB \sin \phi$, or $-x \sin \phi$, since $OB = x$. The projection of BP on the z_w axis is $BP \cos \phi$, or $z \cos \phi$, since $BP = z$. The projection of OP on the z_w axis is z_w. Therefore

$$z_w = -x \sin \phi + z \cos \phi.$$

The equations relating the x, y, z system of axes to the x_w, y_w, z_w system of axes are therefore

$$\begin{aligned} x_w &= x \cos \phi + z \sin \phi. \\ y_w &= y. \\ z_w &= -x \sin \phi + z \cos \phi. \end{aligned}$$

Notice that these equations can be read directly from the box, reading the vertical columns as multiplied by x, y, and z, respectively.

The equations can be solved simultaneously for x, y, and z in terms of x_w, y_w, and z_w to give

$$\begin{aligned} x &= x_w \cos \phi - z_w \sin \phi. \\ y &= y_w. \\ z &= x_w \sin \phi + z_w \cos \phi. \end{aligned}$$

Notice that these equations can be read directly from the box, reading the horizontal rows as multiplied by x_w, y_w, and z_w, respectively. From the box we can read the equations for both

cases: rigged to wing reference plane, and wing reference plane to rigged.

Example 1. The coordinates of a certain point in the wing reference plane system of axes are (100.460, −33.000, 2.787). The angle of dihedral, ϕ, is 4°7′17″. Find the coordinates of this point with respect to the rigged system of axes.

$$\begin{aligned}
\sin\phi &= 0.07187.\\
\cos\phi &= 0.99741.\\
x &= (100.460)(0.99741) - (2.787)(0.07187) = 100.000.\\
y &= -33.000.\\
z &= (100.460)(0.07187) + (2.787)(0.99741) = 10.000.
\end{aligned}$$

The coordinates of the point with respect to the rigged system of axes are therefore (100.000, −33.000, 10.000).

Example 2. The coordinates of a certain point in the rigged system of axes are (150, −70, −60). The angle of dihedral, ϕ, is 4°. Find the coordinates of this point with respect to the wing reference plane system of axes.

$$\begin{aligned}
\sin\phi &= 0.06976.\\
\cos\phi &= 0.99756.\\
x_w &= (150)(0.99756) + (-60)(0.06976) = 145.448.\\
y_w &= -70.\\
z_w &= -(150)(0.06976) + (-60)(0.99756) = -70.318.
\end{aligned}$$

The coordinates of the point with respect to the wing reference plane system of axes are therefore (145.488, −70, −70.318).

Example 3. The equation of a certain plane with respect to the rigged system of axes is

$$3x + 2y - z = 8.$$

The angle of dihedral, ϕ, is 4°. Find the equation of this plane with respect to the wing reference plane system of axes.

$$\begin{aligned}
\sin\phi &= 0.06976.\\
\cos\phi &= 0.99756.\\
x &= 0.99756x_w - 0.06976z_w.\\
y &= y_w.\\
z &= 0.06976x_w + 0.99756z_w.
\end{aligned}$$

Substitute these values for x, y, and z in the given equation of the plane.

$$3(0.99756x_w - 0.06976z_w) + 2y_w - (0.06976x_w + 0.99756z_w) = 8.$$

This result can be simplified to give

$$2.92292x_w + 2y_w - 1.20684z_w = 8.$$

This is the equation of the plane with respect to the wing reference plane system of coordinates.

Example 4. A plane is determined by the three points $A(16, 50, 25)$, $B(10, 32, 15)$, $C(2, 5, 12)$. The coordinates of these points are given with

respect to the wing reference plane system of axes. Find the equation of this plane with respect to the rigged system of axes, if the angle of dihedral, ϕ, is 4°.

$$\begin{aligned}
\sin\phi &= 0.06976.\\
\cos\phi &= 0.99756.\\
x &= (16)(0.99756) - (25)(0.06976) = 14.217.\\
y &= 50.\\
z &= (16)(0.06976) + (25)(0.99756) = 26.055.\\
x &= (10)(0.99756) - (15)(0.06976) = 8.929.\\
y &= 32.\\
z &= (10)(0.06976) + (15)(0.99756) = 15.661.\\
x &= (2)(0.99756) - (12)(0.06976) = 1.158.\\
y &= 5.\\
z &= (2)(0.06976) + (12)(0.99756) = 12.110.
\end{aligned}$$

The coordinates of the three given points with respect to the rigged system of axes are therefore $A(14.217, 50, 26.055)$, $B(8.929, 32, 15.661)$, $C(1.158, 5, 12.110)$. The direction ratios of a normal to the plane are obtained by cross-multiplying the direction ratios of AB and AC:

$$\begin{aligned}
&BA\ \ 5.288:18:10.394.\\
&CA\ \ 13.059:45:13.945.\\
&\text{Normal}\ \ -216.720:61.994:2.898.
\end{aligned}$$

The equation of the plane with respect to the rigged system of axes is

$$-216.720(x - 14.217) + 61.994(y - 50) + 2.898(z - 26.055) = 0.$$

Example 5. The direction cosines of a certain line in the rigged system of axes are 0.99669, 0.08114, 0.00504. Find the direction ratios of this line with respect to the wing reference plane system of axes described in Example 4.

These direction cosines can be treated as if they were the coordinates of a point.

$$\begin{aligned}
x_w &= (0.99669)(0.99756) + (0.00504)(0.06976) = 0.99461.\\
y_w &= 0.08114.\\
z_w &= -(0.99669)(0.06976) + (0.00504)(0.99756) = -0.06450.
\end{aligned}$$

The direction ratios of the line in the wing reference plane system are therefore

$$0.99461:0.08114:-0.06450 \qquad \text{or} \qquad 1:0.08158:-0.06485.$$

If the direction ratios of a line (instead of direction cosines, as in Example 5) are given, then the direction ratios of the same line in another reference system can be calculated by treating the direction ratios as if they were the coordinates of a point, as illustrated in Example 5.

8.4. Rotation of axes (vertical stabilizer and rudder). Sometimes the vertical stabilizer and rudder are so designed that an

angle of rotation is necessary to change the axes from the rigged position to the vertical stabilizer and rudder position. This is often true when the airplane is a single-engine airplane with a single propeller. The rotation helps to counteract propeller torque. An example of this is shown in Fig. 8.7. The rotation takes place about a vertical axis.

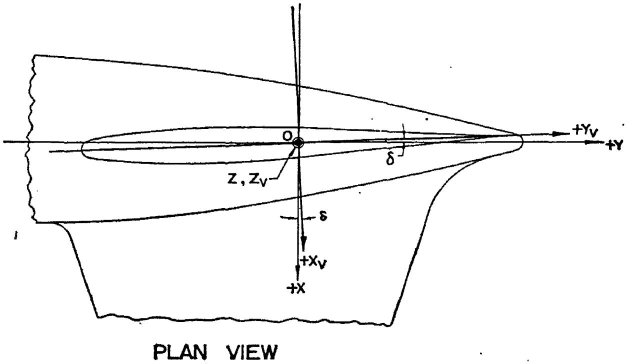


FIG. 8.7.

In Fig. 8.7, the box for the rotation of axes equations is as follows:

	x_v	y_v	z_v
x	$\cos\delta$.	$-\sin\delta$	0
y	$\sin\delta$	$\cos\delta$	0
z	0	0	1

The equations for converting coordinates in the x, y, z system to the x_v, y_v, z_v system are

$$x_v = x\cos\delta + y\sin\delta.$$
$$y_v = -x\sin\delta + y\cos\delta.$$
$$z_v = z.$$

The equations for converting coordinates in the x_v, y_v, z_v system to the x, y, z system are

$$x = x_v\cos\delta - y_v\sin\delta.$$
$$y = x_v\sin\delta + y_v\cos\delta.$$
$$z = z_v.$$

Example 1. A fuselage attachment point has coordinates (12, 18, 15) in the x_v, y_v, z_v system. The angle of rotation, δ, is 1°30′. Find the coordinates of this point in the x, y, z system.

$$
\begin{aligned}
\sin \delta &= 0.02618. \\
\cos \delta &= 0.99966. \\
x &= (12)(0.99966) - (18)(0.02618) = 11.525. \\
y &= (12)(0.02618) + (18)(0.99966) = 18.308. \\
z &= 15.
\end{aligned}
$$

Example 2. A fuselage attachment point has coordinates (18, 25, 50) in the x, y, z system. The angle of rotation, δ, is 1°30′. Find the coordinates of this point in the x_v, y_v, z_v system.

$$
\begin{aligned}
\sin \delta &= 0.02618. \\
\cos \delta &= 0.99966. \\
x_v &= (18)(0.99966) + (25)(0.02618) = 18.648. \\
y_v &= -(18)(0.02618) + (25)(0.99966) = 24.520. \\
z_v &= 50.
\end{aligned}
$$

8.5. Rotation of axes (nacelle). The nacelles are often designed so as to necessitate one angle of rotation from the

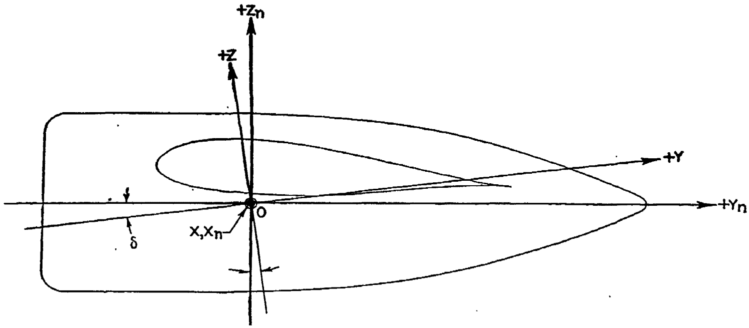


FIG. 8.8.

nacelle system of axes to the rigged system of axes, and vice versa (see Figs. 8.8 and 3.13). The x, y, z system shown in Fig. 8.8 is parallel to the original x, y, z rigged system of axes and is translated to the origin of the nacelle system of axes from the original origin on the plane of symmetry of the airplane. The angle of rotation is designated by δ.

The box for the equations of rotation is

	x_n	y_n	z_n
x	1	0	0
y	0	$\cos\delta$	$\sin\delta$
z	0	$-\sin\delta$	$\cos\delta$

The equations for converting x, y, z coordinates to x_n, y_n, z_n coordinates are

$$x_n = x.$$
$$y_n = y\cos\delta - z\sin\delta.$$
$$z_n = y\sin\delta + z\cos\delta.$$

The equations for converting x_n, y_n, z_n coordinates to x, y, z coordinates are

$$x = x_n.$$
$$y = y_n\cos\delta + z_n\sin\delta.$$
$$z = -y_n\sin\delta + z_n\cos\delta.$$

Example 1. The coordinates of a certain point in the x, y, z system are (10, 20, 30). The angle of rotation, δ, is 2°. Find the coordinates of the point in the x_n, y_n, z_n system.

$$\sin\delta = 0.03490.$$
$$\cos\delta = 0.99939.$$
$$x_n = 10.$$
$$y_n = (20)(0.99939) - (30)(0.03490) = 18.941.$$
$$z_n = (20)(0.03490) + (30)(0.99939) = 30.680.$$

Example 2. The coordinates of a certain point in the x_n, y_n, z_n system are (10, 18.941, 30.680). The angle of rotation, δ, is 2°. Find the coordinates of the point in the x, y, z system.

$$\sin\delta = 0.03490.$$
$$\cos\delta = 0.99939.$$
$$x = 10.$$
$$y = (18.941)(0.99939) + (30.680)(0.03490) = 20.000.$$
$$z = -(18.941)(0.03490) + (30.680)(0.99939) = 30.000.$$

Compare Examples 1 and 2.

Example 3. The direction ratios of a line in the x, y, z system are 1:0.14832:0.52693. The direction ratios of a line in the x_n, y_n, z_n system are 0.21647:1:0.31728. Find the true angle between these two lines, if δ is 2°.

To solve this problem, we must convert the two sets of given direction ratios to the same system of coordinates. Let us choose the x_n, y_n, z_n system.

$$x_n = x.$$
$$y_n = y \cos \delta - z \sin \delta.$$
$$z_n = y \sin \delta + z \cos \delta.$$
$$x_n = 1.$$
$$y_n = (0.14832)(0.99939) - (0.52693)(0.03490) = 0.12984.$$
$$z_n = (0.14832)(0.03490) + (0.52693)(0.99939) = 0.53178.$$

The direction ratios of the two lines are therefore

$$0.21647:1:0.31728,$$
$$1:0.12984:0.53178.$$

The true angle between these two lines is given by

$$\cos \theta = \frac{(0.21647)(1) + (0.12984)(1) + (0.31728)(0.53178)}{\sqrt{(0.21647)^2 + 1^2 + (0.31728)^2}\sqrt{(1^2 + (0.12984)^2 + (0.53178)^2}}$$
$$\cos \theta = 0.42174.$$
$$\theta = 65°3'20''.$$

8.6. Rotation of axes (horizontal stabilizer and elevator). The horizontal stabilizer and elevator are often designed so that one angle of rotation is necessary to rotate the x, y, z system of axes to the x_h, y_h, z_h system of axes and vice versa (see Figs. 8.9 and 3.16).

In Fig. 8.9, the x, y, z system of axes shown is parallel to the rigged system of axes but has its origin translated to the origin of the x_h, y_h, z_h system. The angle θ is an angle of incidence. The box for the rotation of axes equations is

	x_h	y_h	z_h
x	1	0	0
y	0	$\cos \theta$	$\sin \theta$
z	0	$-\sin \theta$	$\cos \theta$

The equations for converting coordinates in the x, y, z system to the x_h, y_h, z_h system are

$$x_h = x.$$
$$y_h = y \cos \theta - z \sin \theta.$$
$$z_h = y \sin \theta + z \cos \theta.$$

The equations for converting coordinates in the x_h, y_h, z_h system to the x, y, z system are

$$x = x_h.$$
$$y = y_h \cos\theta + z_h \sin\theta.$$
$$z = -y_h \sin\theta + z_h \cos\theta.$$

Example. Find the distance from the point (10, 20, 30) in the x, y, z system to the point (6, 5, 12) in the x_h, y_h, z_h system. The angle θ is 2°.

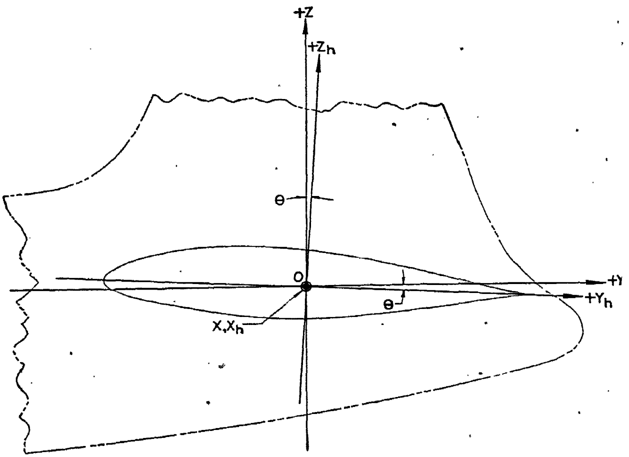


FIG. 8.9.

It is first necessary to convert the coordinates of the two points to the same system of coordinates. Let us choose the x, y, z system.

$$\sin\theta = 0.03490.$$
$$\cos\theta = 0.99939.$$
$$x = 6.$$
$$y = (5)(0.99939) + (12)(0.03490) = 5.416.$$
$$z = -(5)(0.03490) + (12)(0.99939) = 11.818.$$

Next use the length of a line segment formula

$$L = \sqrt{(10-6)^2 + (20-5.416)^2 + (30-11.818)^2}.$$
$$L = 23.649.$$

This is the required distance between the two points.

8.7. Rotation of axes (wing chord plane). When a wing is lofted by the wing chord plane system, two angles of rotation are necessary to change the rigged system of axes to the wing chord plane system of axes, and vice versa. Since the origin is usually the same in both the rigged and wing chord plane systems, no translation of axes is necessary (see Figs. 8.10, 3.9, 3.10, 3.11, and 5.2).

The projection of OP on the x_w axis is equal to the sum of the projections of OA on the x_w axis, AB on the x_w axis, and BP on the x_w axis, since OP is the closing line segment of the broken line

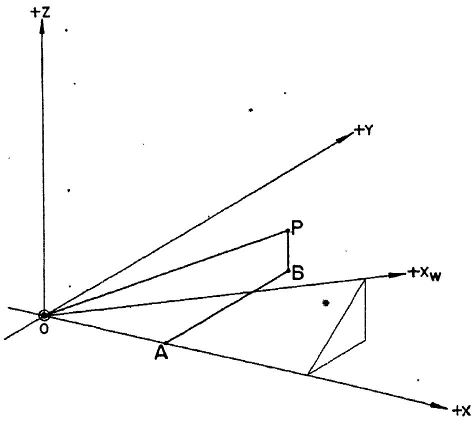


FIG. 8.10.

$OABP$. The projection of OA on the x_w axis is $x \cos \phi$, since $\cos \phi$ is the cosine of the true angle between the x axis and the x_w axis (refer to Art. 5.3). The projection of AB on the x_w axis is $y \sin \phi \sin \theta$, since $\sin \phi \sin \theta$ is the cosine of the true angle between the y axis and the x_w axis. The projection of BP on the x_w axis is $z \sin \phi \cos \theta$, since $\sin \phi \cos \theta$ is the cosine of the true angle between the z axis and the x_w axis. Therefore

$$x_w = x \cos \phi + y \sin \phi \sin \theta + z \sin \phi \cos \theta.$$

Notice that this formula for x_w may be obtained by multiplying the x, y, z by the direction cosines in the first vertical column in the box:

	x_w	y_w	z_w
x	$\cos\phi$	0	$-\sin\phi$
y	$\sin\phi\sin\theta$	$\cos\theta$	$\cos\phi\sin\theta$
z	$\sin\phi\cos\theta$	$-\sin\theta$	$\cos\phi\cos\theta$

Formulas for y_w and z_w can be derived on the basis of the projection theorems in a similar way. The results can be read directly from the above box.

The equations for converting coordinates in the x, y, z system to the x_w, y_w, z_w system are

$$
\begin{aligned}
x_w &= x\cos\phi + y\sin\phi\sin\theta + z\sin\phi\cos\theta,\\
y_w &= y\cos\theta - z\sin\theta,\\
z_w &= -x\sin\phi + y\cos\phi\sin\theta + z\cos\phi\cos\theta.
\end{aligned}
$$

The equations for converting coordinates in the x_w, y_w, z_w system to the x, y, z system are

$$
\begin{aligned}
x &= x_w\cos\phi - z_w\sin\phi,\\
y &= x_w\sin\phi\sin\theta + y_w\cos\theta + z_w\cos\phi\sin\theta,\\
z &= x_w\sin\phi\cos\theta - y_w\sin\theta + z_w\cos\phi\cos\theta.
\end{aligned}
$$

These two sets of equations offer a compact solution to a problem that is ordinarily quite troublesome. They enable us to locate points with respect to the basic reference planes of the fuselage when the points are known with respect to the basic reference planes of a wing lofted by the chord plane system, and vice versa. The angle ϕ represents the dihedral angle and the angle θ represents the angle of incidence. In problems involving landing gears, attach fittings connecting wing and fuselage, fillets connecting wing and fuselage, and in many other situations it is necessary to be able to convert coordinates from the wing chord plane system to the fuselage reference system (rigged system), and vice versa.

Example. A point has coordinates (100, 50, 10) in the wing chord plane system. The angle of dihedral is 3° and the angle of incidence is 2°. Find the coordinates of the point in the rigged system of axes.

$\sin 2^\circ = 0.03490. \qquad \sin 3^\circ = 0.05234.$
$\cos 2^\circ = 0.99939. \qquad \cos 3^\circ = 0.99863.$

$$x = (100)(0.99863) - (10)(0.05234) = 99.340.$$

$$y = (100)(0.05234)(0.03490) + (50)(0.99939) + (10)(0.99863)(0.03490) = 50.501.$$

$$z = (100)(0.99939)(0.05234) - (50)(0.03490) + (10)(0.99939)(0.99863) = 13.466.$$

The coordinates of the point in the rigged system are (99.340, 50.501, 13.466).

A good method of checking the results is to use the equations for x_w, y_w, z_w in terms of x, y, z and determine whether the answer is the set of given coordinates (100, 50, 10). A partial check is to find the square root of the sum of the squares of the calculated coordinates. This result should equal the square root of the sum of the squares of the original coordinates. That is,

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{x_w^2 + y_w^2 + z_w^2}.$$

This relation is due to the fact that each side of this last equation is equal to the distance from the origin to the point, and a distance is invariant under rotation of axes.

The remarks in the last paragraph apply equally well to all cases of rotations considered in this chapter.

8.8. Translation and rotation of axes. In most cases it is necessary both to translate and to rotate the axes. An exception to this is the case described in the preceding article. See Fig. 8.11. In Fig. 8.11, the original set of axes is the x, y axes. The original set of axes is first translated to a new origin at O', resulting in the x', y' set of axes. The x', y' set of axes is rotated through an angle θ to give the final x_a, y_a set of axes.

Example 1. In Fig. 8.11, let the coordinates of O' with respect to the x, y system of axes be (10, 8). Also, let the angle of rotation, θ, be 3°. If a certain point P has coordinates (15, 40) with respect to the x, y system of axes, what are its coordinates with respect to the x_a, y_a set of axes?

$$= x - 10,$$
$$= y - 8.$$
$$= 15 - 10 = 5,$$
$$= 40 - 8 = 32.$$
$$= x' \cos \theta + y' \sin \theta,$$
$$= -x' \sin \theta + y' \cos \theta.$$
$$= (5)(0.99863) + (32)(0.05234) = 6.668,$$
$$= (-5)(0.05234) + (32)(0.99863) = 31.694.$$

Therefore the coordinates of the given point with respect to the x_a, y_a set of axes are (6.668, 31.694).

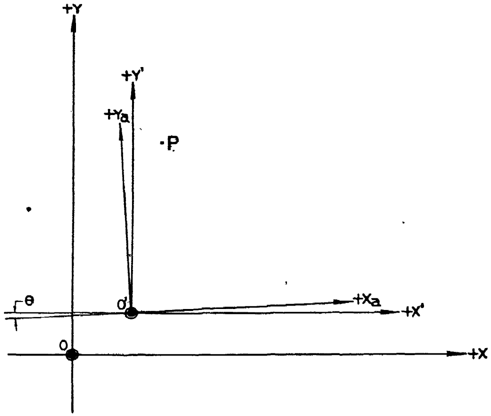


FIG. 8.11.

Example 2. In Fig. 8.12, the point P has coordinates (64.312, 20.569, 25.782) with respect to the x, y, z system of coordinates. The angle ϕ is 7°

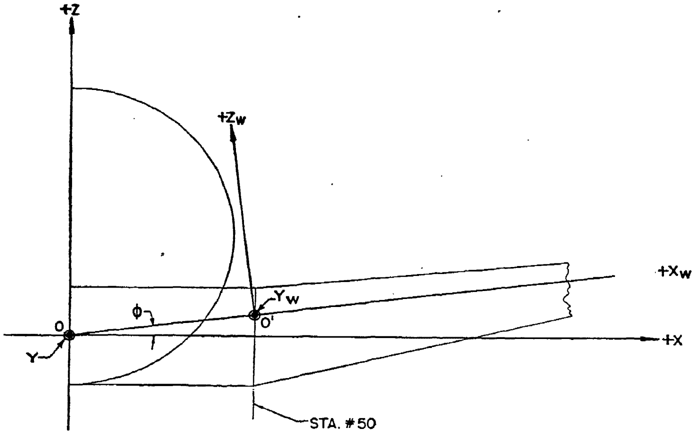


FIG. 8.12.

and O' is (50, 0, 6.139). Find the coordinates of P with respect to the x_w, y_w, z_w system of coordinates.

$$x_w = (x - 50)\cos\phi + (z - 6.139)\sin\phi.$$
$$y_w = y.$$

$$z_w = -(x - 50) \sin \phi + (z - 6.139) \cos \phi.$$
$$x_w = (14.312)(0.99255) + (19.643)(0.12187) = 16.599.$$
$$y_w = 20.569.$$
$$z_w = (-14.312)(0.12187) + (19.643)(0.99255) = 17.752.$$

The coordinates of P with respect to the x_w, y_w, z_w system of axes are therefore (16.599, 20.569, 17.752).

8.9. Rotation of axes (special cases). There are certain special cases of rotating axes which are of extreme importance.

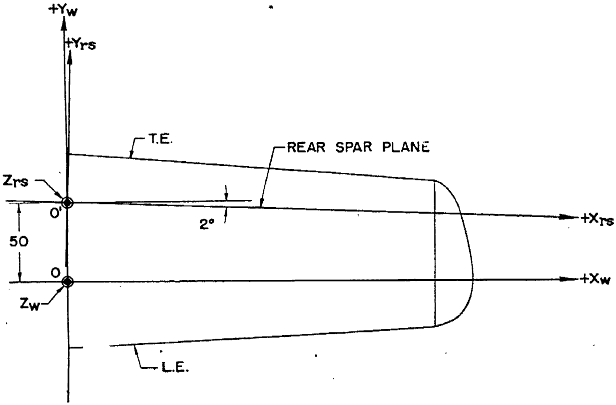


FIG. 8.13.

These special cases can be treated by techniques similar to those used in the preceding articles of this chapter.

Consider the case of a rear spar plane as shown on an engineering drawing of a rear spar. We can set up a set of axes for the rear spar plane and label them x_{rs}, y_{rs}, z_{rs}. Their relation to the wing reference plane axes is shown in Fig. 8.13. The box relating the two sets of axes is

	x_{rs}	y_{rs}	z_{rs}	O'
x_w	cos 2°	sin 2°	0	0
y_w	− sin 2°	cos 2°	0	50
z_w	0	0	1	0
O	1.7450	−49.9695	0	

Notice that a vertical column has been added that gives the x_w, y_w, z_w coordinates of the origin of the x_{rs}, y_{rs}, z_{rs} system of axes as $O'(0, 50, 0)$. Also, a horizontal row has been added that gives the x_{rs}, y_{rs}, z_{rs} coordinates of the origin of the x_w, y_w, z_w system of axes as $O(1.7450, -49.9695, 0)$. The first set of coordinates is obvious from Fig. 8.13, but the second set of coordinates must be calculated using the rotation of axes equations

$$
\begin{aligned}
x_{rs} &= x_w \cos 2^\circ - y_w \sin 2^\circ. \\
y_{rs} &= x_w \sin 2^\circ + y_w \cos 2^\circ. \\
z_{rs} &= z_w.
\end{aligned}
$$

$$
\begin{aligned}
x_{rs} &= (0)(0.99939) - (-50)(0.03490) = 1.7450. \\
y_{rs} &= (0)(0.03490) + (-50)(0.99939) = -49.9695. \\
z_{rs} &\quad 0.
\end{aligned}
$$

This enlarged box makes the matter of converting the coordinates of a point from one reference system to another a simple process. This enlarged box is especially useful when a large number of sets of coordinates must be converted from one system to another. It simplifies the calculations.

However, when the origins of the two systems of axes are coincident, the added row and column are not necessary. Also, when converting a set of direction ratios, or direction cosines, from one system to another, the enlarged box is not used.

The equations converting coordinates in the x_{rs}, y_{rs}, z_{rs} system to the x_w, y_w, z_w system are

$$
\begin{aligned}
x_w &= x_{rs} \cos 2^\circ + y_{rs} \sin 2^\circ, \\
y_w &= -x_{rs} \sin 2^\circ + y_{rs} \cos 2^\circ + 50, \\
z_w &= z_{rs}.
\end{aligned}
$$

The equations converting coordinates in the x_w, y_w, z_w system to the x_{rs}, y_{rs}, z_{rs} system are

$$
\begin{aligned}
x_{rs} &= x_w \cos 2^\circ - y_w \sin 2^\circ + 1.7450, \\
y_{rs} &= x_w \sin 2^\circ + y_w \cos 2^\circ - 49.9695, \\
z_{rs} &= z_w.
\end{aligned}
$$

Example 1. A point on the top lofted line of the rear spar has coordinates (50, 2, 10) in the x_{rs}, y_{rs}, z_{rs} system. Find its coordinates in the x_w, y_w, z_w system (see Fig. 8.13).

$$x_w = (50)(0.99939) + (2)(0.03490).$$
$$y_w = -(50)(0.03490) + (2)(0.99939) + 50.$$
$$z_w = 10.$$
$$x_w = 50.039.$$
$$y_w = 50.254.$$
$$z_w = 10.$$

Example 2. Find the coordinates of the point in Example 1 with respect to the rigged system of axes, if the angle of dihedral is 4°7′17″.

We have already converted the point from the rear spar subassembly position to the wing reference plane system. Now we use the equations for converting coordinates from the wing reference plane system to the rigged system.

$$x = 0.99741x_w - 0.07187z_w,$$
$$y = y_w,$$
$$z = 0.07187x_w + 0.99741z_w.$$
$$x = (50.039)(0.99741) - (10)(0.07187) = 49.191.$$
$$y = 50.254.$$
$$z = (50.039)(0.07187) + (10)(0.99741) = 13.570.$$

Example 3. Find the direction ratios of a normal to the rear spar plane in Fig. 8.13 with respect to the rigged system of axes. Use $\phi = 4°7'17''$.

The rear spar plane is the $x_{rs}z_{rs}$ plane. The y_{rs} axis is normal to this plane. The direction ratios of the y_{rs} axis in the rear spar plane system are 0:1:0. Treat these three numbers as if they were the coordinates of a point, and convert them to the wing reference plane system by the equations for rotating axes. We obtain

$$x_w = \sin 2° = 0.03490,$$
$$y_w = \cos 2° = 0.99939,$$
$$z_w = 0.$$

Now use the rotation of axes equations for converting coordinates from the wing reference plane system to the rigged system.

$$x = (0.03490)(0.99741) - (0)(0.07187) = 0.03481.$$
$$y = 0.99939.$$
$$z = (0.03490)(0.07187) + (0)(0.99741) = 0.00251.$$

The direction ratios of a normal to the rear spar plane with respect to the rigged system of axes are 0.03481:0.99939:0.00251 or 0.03483:1:0.00251.

Example 4. Consider the case of an engine mount that has been rotated about an axis parallel to the y axis of the rigged system of axes. In this case there is a subassembly set of axes for the engine mount, which are related to the rigged system of axes as shown in Fig. 8.14. In Fig. 8.14, the x, y, z system is the rigged system of axes, and the x_m, y_m, z_m is the engine mount system of axes. Both translation and rotation are necessary to move the rigged system to the position of the engine mount system, and vice versa.

A point on the engine mount has coordinates (40, 65, 30) with respect to the engine mount system of axes. Find its coordinates with respect to the rigged system of axes.

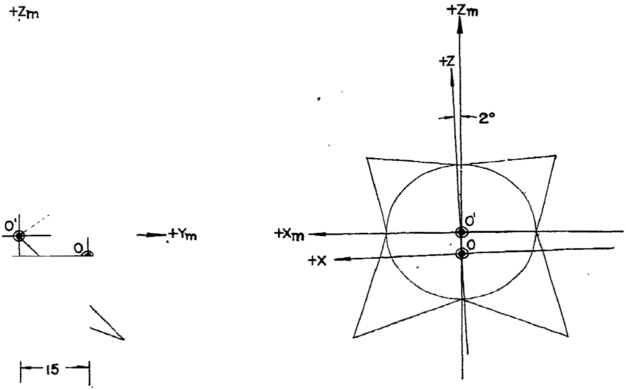


FIG. 8.14.

The box relating the two sets of axes for purposes of rotation only is

	x	y	z
x_m	$\cos 2°$	0	$\sin 2°$
y_m	0	1	0
z_m	$-\sin 2°$	0	$\cos 2°$

The equations for converting coordinates from the x_m, y_m, z_m system to the x, y, z system are

$$\begin{aligned} x &= 0.99939x_m - 0.03490z_m, \\ y &= y_m, \\ z &= 0.03490x_m + 0.99939z_m. \end{aligned}$$

Correcting first for translation, the new coordinates of the point are (40, 50, 35). Substituting the values $x_m = 40$, $y_m = 50$, and $z_m = 35$,

$$\begin{aligned} x &= (0.99939)(40) - (0.03490)(35) = 38.754. \\ y &= 50. \\ z &= (0.03490)(40) + (0.99939)(35) = 36.375. \end{aligned}$$

The coordinates of the point in the rigged system of axes are (38.754, 50, 36.375).

Sometimes the x, y_w coordinates of a point are known and it is necessary to find the x_w, y coordinates of the point. This can be done by solving two linear equations simultaneously.

Example 5. Consider a point in a wing lofted by the wing reference plane system. For this point $x = 100.000$ and $z_w = 2.787$. Find x_w and z, where x and z refer to the rigged system, as usual. The angle of dihedral is 4°7′17″.

Recall that $y = y_w$, so the y coordinate is not affected in this case.

$$\begin{aligned}
x &= x_w \cos \phi - z_w \sin \phi, \\
z &= x_w \sin \phi + z_w \cos \phi. \\
100.000 &= 0.99741x_w - (0.07187)(2.787), \\
z &= 0.07187x_w + (0.99741)(2.787).
\end{aligned}$$

Solving these two equations for x_w and z, we obtain

$$\begin{aligned}
x_w &= 100.460, \\
z &= 10.000.
\end{aligned}$$

8.10. Combined rotations. It is often convenient to combine two successive rotations. Suppose that the wing reference plane coordinates of a certain point are (x_w, y_w, z_w), and suppose that the equations for converting coordinates from the wing reference plane system of axes to the rigged system of axes are

$$\begin{aligned}
x &= x_w \cos \phi - z_w \sin \phi, \\
y &= y_w, \\
z &= x_w \sin \phi + z_w \cos \phi.
\end{aligned}$$

Furthermore, suppose that the equations for converting coordinates from the rigged system of axes to the nacelle system of axes are

$$\begin{aligned}
x_n &= x, \\
y_n &= y \cos \delta - z \sin \delta, \\
z_n &= y \sin \delta + z \cos \delta.
\end{aligned}$$

Under these circumstances, we can convert coordinates from the wing reference plane system of axes to the nacelle system of axes in two successive steps. However, an alternative method is to combine the two steps into one step.

$$\begin{aligned}
x_n &= x_w \cos \phi - z_w \sin \phi, \\
y_n &= y_w \cos \delta - x_w \sin \phi \sin \delta - z_w \cos \phi \sin \delta, \\
z_n &= y_w \sin \delta + x_w \sin \phi \cos \delta + z_w \cos \phi \cos \delta.
\end{aligned}$$

With these equations we can convert coordinates from the wing

reference plane system of axes to the nacelle system of axes in one step.

Example 1. Let $\phi = 4°7'17''$ and $\delta = 2°$. Find the equations for converting coordinates from the wing reference plane system of axes to the nacelle system of axes.

$$\sin 4°7'17'' = 0.07187. \qquad \sin 2° = 0.03490.$$
$$\cos 4°7'17'' = 0.99741. \qquad \cos 2° = 0.99939.$$

Substituting these values in the preceding set of equations,

$$\begin{aligned} x_n &= 0.99741x_w - 0.07187z_w. \\ y_n &= -0.00251x_w + 0.99939y_w - 0.03481z_w. \\ z_n &= 0.07183x_w + 0.03490y_w + 0.99680z_w. \end{aligned}$$

Example 2. In Example 1, let the coordinates of a certain point be

$$\begin{aligned} x_w &= 100.460. \\ y_w &= -33.000. \\ z_w &= 2.787. \end{aligned}$$

Find the coördinates of this point with reference to the x_n, y_n, z_n system of axes.

$$\begin{aligned} x_n &= (0.99741)(100.460) - (0.07187)(2.787) - 100.000. \\ y_n &= (-0.00251)(100.460) + (0.99939)(-33.000) - (0.03481)(2.787) \\ &= -33.329. \\ z_n &= (0.07183)(100.460) + (0.03490)(-33.000) + (0.99680)(2.787) \\ &= 8.842. \end{aligned}$$

A good method of tabulating the equations is to arrange the values in a box:

	x_w	y_w	z_w
x_n	0.99741	0	−0.07187
y_n	−0.00251	0.99939	−0.03481
z_n	0.07183	0.03490	0.99680

From this box we can also write the equations converting x_n, y_n, z_n coordinates into x_w, y_w, z_w coördinates:

$$\begin{aligned} x_w &= 0.99741x_n - 0.00251y_n + 0.07183z_n, \\ y_w &= 0.99939y_n + 0.03490z_n, \\ z_w &= -0.07187x_n - 0.03481y_n + 0.99680z_n. \end{aligned}$$

A good method of checking the values in the box is to square the set of values in each horizontal row and add. The result should be one in each case. A similar check holds for the vertical columns.

If the origin of the nacelle system of axes is different from the origin of the rigged or wing reference plane system of axes, then translation must be taken into account when converting coordinates from the rigged system of axes to the nacelle system of axes. Sometimes translation comes before rotation. This is the case when the origin of the nacelle system of axes is given in terms of rigged coordinates. For example, if the origin of the nacelle system of axes has coordinates $x = 75.566$, $y = 0.628$, $z = -12.533$ with respect to the rigged system of axes, and if the coordinates of a certain point are $x = 100$, $y = -33$, $z = 10$ with respect to the rigged system of axes, then the nacelle coordinates of the point can be found in two steps:

$$\begin{aligned}(a)\quad 100 - 75.566 &= 24.434.\\ -33 - 0.628 &= -33.628.\\ 10 + 12.533 &= 22.533.\end{aligned}$$

$$\begin{aligned}(b)\quad x_n &= 24.434.\\ y_n &= (-33.628)(0.99939) - (22.533)(0.03490) = -34.394.\\ z_n &= (-33.628)(0.03490) + (22.533)(0.99939) = 21.346.\end{aligned}$$

If the origin of the rigged or wing reference plane system of axes is given in nacelle coordinates, translation follows rotation.

8.11. General remarks on rotation of axes. In each case of rotation of axes we used a box of the type

	x	y	z
x'	a	b	c
y'	d	e	f
z'	g	h	i

Certain relations exist among the sets of direction cosines in such a box. Some of these are

$$\begin{aligned}a^2 + b^2 + c^2 &= 1. &\qquad a^2 + d^2 + g^2 &= 1.\\ d^2 + e^2 + f^2 &= 1. & b^2 + e^2 + h^2 &= 1.\\ g^2 + h^2 + i^2 &= 1. & c^2 + f^2 + i^2 &= 1.\end{aligned}$$

These formulas are true because the sum of the squares of the direction cosines must equal one. Also,

$$ad + be + cf = 0. \qquad ab + de + gh = 0.$$
$$ag + bh + ci = 0. \qquad ac + df + gi = 0.$$
$$dg + eh + fi = 0. \qquad bc + ef + hi = 0.$$

These formulas are true because the sum of the products of corresponding direction cosines of two perpendicular lines must equal the cosine of 90°, which is 0. Also,

$$bf - ce = g.$$
$$cd - af = h.$$
$$ae - bd = i.$$

These formulas are true because they represent the usual method of cross-multiplying to find the direction ratios of a line perpendicular to two given lines. If the first two horizontal rows of direction cosines are known, the direction cosines of the bottom horizontal row can be calculated. Similar relations hold for the direction cosines of any row or column.

CHAPTER 9

APPLICATIONS

This chapter deals with applications of the theory developed in the previous chapters. Some of the illustrations apply to isolated problems, some to descriptive geometry, and some to an actual wing, designed in some detail.

It is impossible to give a complete survey of the possibilities of the applications of solid analytic geometry to the airplane. The examples in preceding chapters and in this chapter are typical and indicate the tremendous scope and power of this mathematical tool. An attempt is made in this chapter to tie together the various isolated principles developed in the preceding chapters.

In applying the concepts and methods of solid analytic geometry to the airplane, it is essential to make full use of calculating machine techniques in making computations orderly, systematic, and economical of time and space. Special attention must also be given to the clear concise display of calculated and tabulated data.

In this chapter we explain the preparation and use of

1. Basic data tables.
2. Master diagram.
3. Loft layout.

Upon these three items the lofting and basic calculations groups can base their proper functions with relation to preliminary design, engineering, and tooling. The basic data tables, master diagram, and loft layout used in this chapter refer to a wing designed especially for this chapter.

9.1. Cylinder and planes. In Fig. 9.1 the center line of the tube is a skew (canted) line in space. A clamp is to be attached to the tube. The plane of the clamp is normal to the xy plane in the plan view and is normal to the xz plane in the front view, as shown in Fig. 9.1. Find the true angle between the plane of the clamp and the center line of the tube.

Set up an auxiliary system of axes, as shown. The plane of the clamp is the yz plane. The x axis is normal to the yz plane. Assume a unit distance on the x axis, as shown. The direction ratios of the center line of the tube are $1:\tan 19°:\tan 5°$. Com-

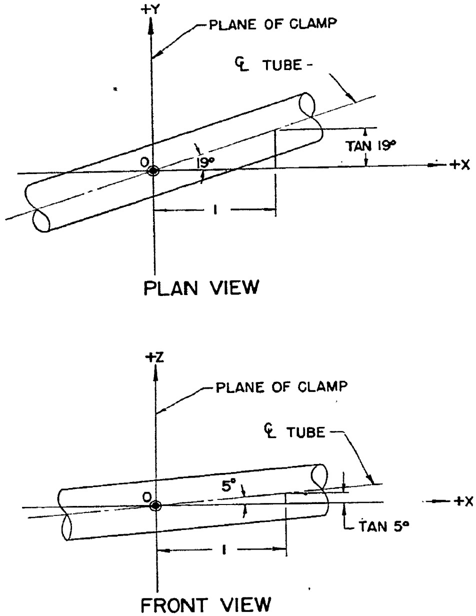


FIG. 9.1.

pute the first direction cosine of the center line of the tube. It is

$$\cos \alpha \doteq \frac{1}{\sqrt{1 + \tan^2 19° + \tan^2 5°}}.$$

$$\cos \alpha = 0.94230.$$

By the definition of direction cosines, this is the cosine of the true angle between the center line of the tube and the x axis. But the x axis is normal to the plane of the clamp, and the angle between a line and a plane is the complement of the angle between the line and a normal to the plane. Therefore the angle between the center line of the tube and the plane of the clamp is

the complement of α. Since $\cos \alpha = \sin (90° - \alpha)$ we can find the required angle by finding α in the sine tables, instead of the cosine tables. The result is 70°26′30″.

This example is typical of the general problem of calculating the true angle between a center line of a tube and a plane that

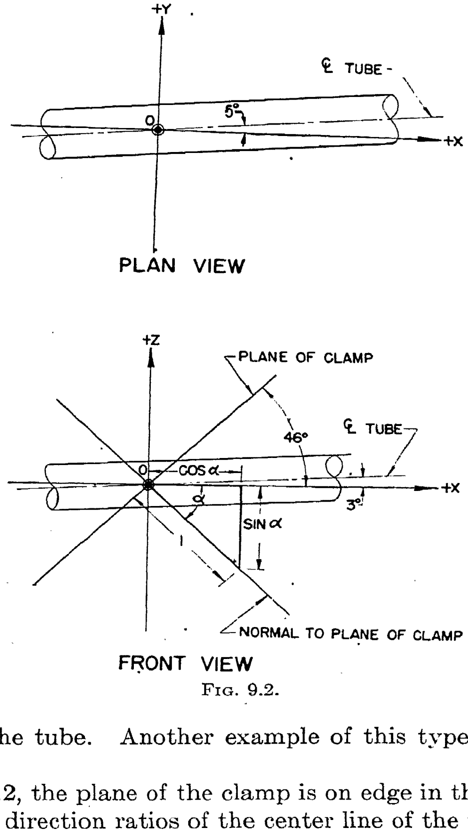


FIG. 9.2.

intersects the tube. Another example of this type is shown in Fig. 9.2.

In Fig. 9.2, the plane of the clamp is on edge in the front view only. The direction ratios of the center line of the tube are

$$1 : \tan 5° : \tan 3°.$$

The direction cosines of a normal to the plane of the clamp are

$$\cos \alpha,\ 0,\ -\sin \alpha \qquad \text{or} \qquad 0.71934,\ 0,\ -0.69466,$$

where $\alpha = 90° - 46° = 44°$. Reduce the direction ratios of the center line of the tube to direction cosines. We obtain

$$0.99484, 0.08704, 0.05214.$$

The true angle between the center line of the tube and the plane of the clamp is given by

$$\sin \theta = 0.67941.$$
$$\theta = 42°47'51''.$$

If the plane of the clamp is not on edge in any of the three basic orthographic views, then the plane can be determined by

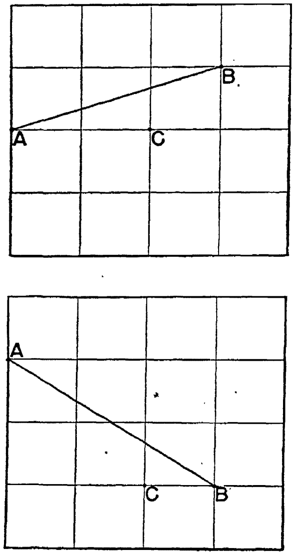


FIG. 9.3.

three points, and the direction ratios of a normal to the plane can be determined by cross-multiplying the direction ratios of any two lines in the plane. Then the true angle between the center line of the tube and the normal can be calculated. The required angle will be the complement of this latter result.

9.2. Descriptive geometry textbook problems. Descriptive geometry textbooks state problems in one of two ways. Either

a drawing is given, with dimensions as in Figs. 9.1 and 9.2, or a graph-paper coordinate system is used to locate points, lines, and planes. In the latter case, solid analytic geometry can be applied directly to check the solution by layout procedure. As an example of this type, see Fig. 9.3.

In Fig. 9.3 the line AB and the point C are given. It is required to find the distance from the point to the line. Set up a system of axes, as shown in Fig. 9.4.

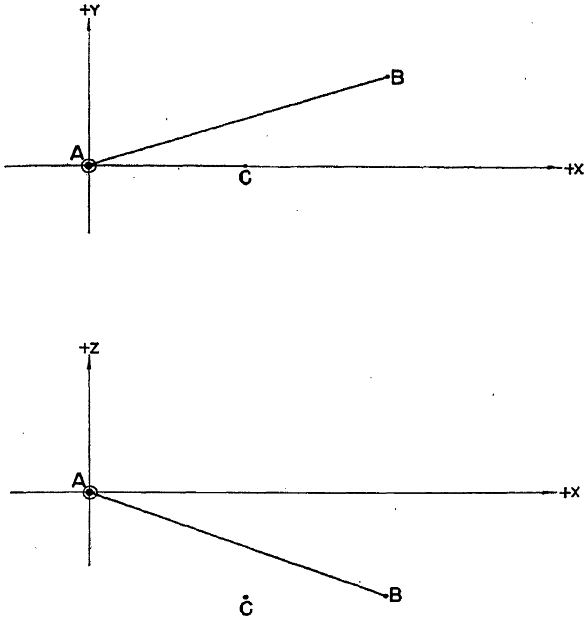


FIG. 9.4.

The coordinates of the three points are

$$A\ (0,\ 0,\ 0),$$
$$B\ (3,\ 1,\ -2),$$
$$C\ (2,\ 0,\ -2).$$

The direction ratios of AB are $3:1:-2$. The direction cosines of AB are $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$. The direction ratios of CB are

1:1:0. The direction cosines of CB are $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$. The true angle between AB, CB is given by

$$\cos\theta = \frac{1}{\sqrt{14}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{14}}\frac{1}{\sqrt{2}} - \frac{}{\sqrt{28}}.$$
$$\cos\theta = 0.75593.$$
$$\theta = 40°53'35''.$$

The true length of the line segment CB is $\sqrt{2}$. The distance

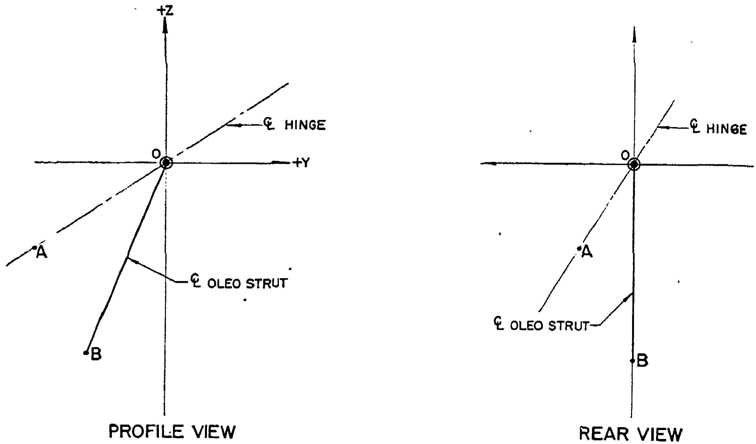


FIG. 9.5.

from C to AB is given by

$$d = \sqrt{2}\sin\theta,$$
$$d = 0.926.$$

This example is typical of the usual descriptive geometry textbook problem. Both students and instructors can use the method of solid analytic geometry to check such problems.

9.3. A landing-gear problem. Sometimes a landing gear is so designed that the center line of oleo strut revolves about an imaginary hinge center line (see Fig. 9.5). The center line of hinge is the axis of a right circular cone. The line OB rotates about this axis at O. The true distance from B to the line OA is the radius of the base of this cone. The axes in rigged position are x, y, z, in Fig. 9.5. To study the rotation of OB about OA, it

is convenient to set up a new system of axes with OA as the new x_s axis. The lines OA, OB determine a plane. Let the new y_s axis be normal to this plane, and let the new z_s axis be normal to both the x_s axis and the y_s axis (see Fig. 9.6). The problem reduces to the matter of determining the rotation of axes equations from the x, y, z system to the x_s, y_s, z_s system, and vice versa.

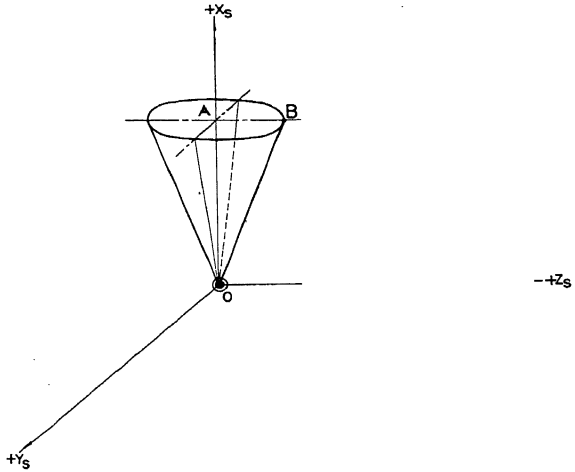


FIG. 9.6.

The steps in determining these equations are as follows:

1. Calculate the direction cosines of OA, and label them a, b, c.
2. Calculate the direction cosines of OB, and label them d, e, f.
3. Calculate the direction cosines of a normal to OA and OB, and label them g, h, i.
4. Calculate the direction cosines of a normal to OA and the result of step 3, and label them p, q, r.
5. Arrange the box:

	x_s	y_s	z_s
x	a	g	p
y	b	h	q
z	c	i	r

The rotation of axes equations are

$$\begin{aligned} x &= ax_s + gy_s + pz_s, \\ y &= bx_s + hy_s + qz_s, \\ z &= cx_s + iy_s + rz_s, \end{aligned}$$

and

$$\begin{aligned} x_s &= ax + by + cz, \\ y_s &= gx + hy + iz, \\ z_s &= px + qy + rz. \end{aligned}$$

With these equations it is possible to convert the coordinates of B from the x_s, y_s, z_s system to the x, y, z system, and vice versa. It is therefore possible to study the kinematics of the landing gear.

This problem can also be applied to the revolution of a point about a line, the point being B and the line being OA. This situation arises quite frequently in tool design. Although the center line of oleo strut is shown to be vertical in the rear view in Fig. 9.5, the method explained is quite general and applies equally well to the case in which OB is a canted (skew) line in space. In landing-gear problems it is necessary to make a careful distinction between rigged dimensions and wing reference plane (or wing chord plane) dimensions. Some dimensions of retracting landing gears are with respect to the fuselage reference system of axes, and others are with respect to the wing system of axes. The engineering drawings must therefore be consulted carefully and thoroughly, and, when necessary, all coordinates must be converted to one of the two systems.

9.4. Special planes. There are certain fundamental planes in the wing and in the fuselage which are basic in rigging the wing. Some of these planes are vertical rib planes, fuselage station planes, the horizontal reference plane, normal rib planes, and the wing reference plane (or the wing chord plane, depending upon the method of rigging the wing). It is necessary to know the true angles between these various planes (refer to Figs. 3.5, 3.10, and 3.12).

Example 1. Find the true angle between a fuselage station plane (rigged position) and a normal rib plane, when the wing is rigged by the wing chord plane system (see Fig. 3.10).

The x_w axis is perpendicular to the normal rib planes. The direction cosines of the x_w axis are 1, 0, 0. The y axis is perpendicular to the fuselage

station planes. The direction cosines of the y axis are 0, 1, 0. These direction cosines must be converted to the wing chord plane system of axes. To do this, refer to the box showing the relations between the wing chord plane system of axes and the rigged system of axes. The formulas for con-

WING REFERENCE PLANE

RIGGED POSITION OF PLANES	NORMAL RIB PLANE	WING REFERENCE PLANE
VERTICAL RIB PLANE	$\cos A = \cos \phi$	$\cos A = -\sin \phi$
FUS. STA. PLANE	$\cos A = 0$	$\cos A = 0$
HORIZ. REF. PLANE	$\cos A = \sin \phi$	$\cos A = \cos \phi$

ϕ = ANGLE OF DIHEDRAL

A = ANGLE BETWEEN PLANES

FIG. 9.7.

WING CHORD PLANE

RIGGED POSITION OF PLANES	NORMAL RIB PLANE	CHORD PLANE
VERTICAL RIB PLANE	$\cos A = \cos \phi$	$\cos A = -\sin \phi$
FUS STA. PLANE	$\cos A = \sin \phi \sin \theta$	$\cos A = \cos \phi \sin \theta$
HORIZ. REF PLANE	$\cos A = \sin \phi \cos \theta$	$\cos A = \cos \phi \cos \theta$

ϕ = ANGLE OF DIHEDRAL

θ = ANGLE OF INCIDENCE

A = ANGLE BETWEEN PLANES

FIG. 9.8.

verting x, y, z coordinates to x_w, y_w, z_w coordinates are

$$\begin{aligned} x_w &= x \cos \phi + y \sin \phi \sin \theta + z \sin \phi \cos \theta, \\ y_w &= y \cos \theta - z \sin \theta, \\ z_w &= -x \sin \phi + y \sin \theta \cos \phi + z \cos \phi \cos \theta. \end{aligned}$$

Now $x = 0$, $y = 1$, $z = 0$ for the y axis. Therefore $x_w = \sin \phi \sin \theta$, $y_w = \cos \theta$, $z_w = \sin \theta \cos \phi$. We now have the direction cosines of a normal to the normal rib plane and the direction cosines of a normal to the fuselage station plane, both in the wing chord plane system. They are

$$1, 0, 0,$$
$$\sin \phi \sin \theta, \cos \theta, \sin \theta \cos \phi.$$

WING REFERENCE PLANE SYSTEM

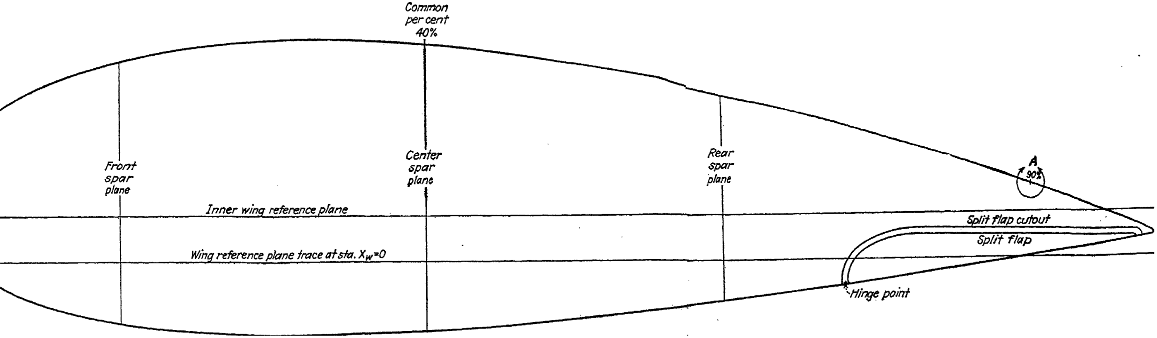
	Wing reference plane position	Rigged position
Normal rib plane......	1:0:0	cos ϕ:0:sin ϕ
Wing reference plane...	0:0:1	$-$ sin ϕ:0:cos ϕ
Vertical rib plane......	cos ϕ:0: $-$ sin ϕ	1:0:0
Plane of symmetry.....	cos ϕ:0: $-$ sin ϕ	1:0:0
Water line plane.......	sin ϕ:0:cos ϕ	0:0:1
Buttock line plane	cos ϕ:0: $-$ sin ϕ	1:0:0
Fuselage station plane..	0:1:0	0:1:0
Rear spar plane........	sin α:cos α:0	tan α cos ϕ:1:tan α sin ϕ
Front spar plane.......	sin β: $-$ cos β:0	$-$ tan β cos ϕ:1: $-$ tan β sin ϕ
Common per cent plane.	0:1:0	0:1:0

NOTE: These are direction cosines, with the exception of the normals to the spar planes in rigged position.

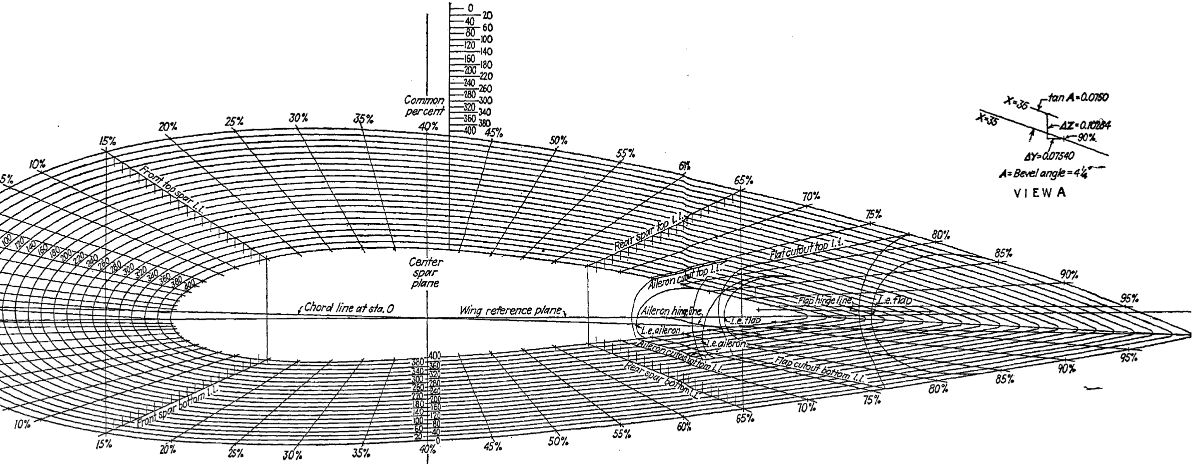
WING CHORD PLANE SYSTEM

	Wing chord plane position	Rigged position
Normal rib plane......	1:0:0	cos ϕ:sin ϕ sin θ:sin ϕ cos θ
Wing chord plane......	0:0:1	$-$sin ϕ:cos ϕ sin θ: cos ϕ cos θ
Vertical rib plane......	cos ϕ:0: $-$ sin ϕ	1:0:0
Plane of symmetry.....	cos ϕ:0: $-$ sin ϕ	1:0:0
Water line plane.......	sin ϕ cos θ: $-$ sin θ: cos ϕ cos θ	0:0:1
Buttock line plane.....	cos ϕ:0: $-$ sin ϕ	1:0:0
Fuselage station plane..	sin ϕ sin θ:cos θ:cos ϕ sin θ	0:1:0
Rear spar plane........	sin α:cos α:0	tan α cos ϕ:tan α sin ϕ sin θ + cos θ:tan α sin ϕ cos θ $-$ sin θ
Front spar plane.......	sin β: $-$ cos β:0	tan β cos ϕ:tan β sin ϕ sin θ $-$ cos θ:tan β sin ϕ cos θ + sin θ
Common per cent plane.	0:1:0	0:cos θ: $-$ sin θ

NOTE: These are direction cosines, with the exception of the normals to the spar planes in rigged position.



VERTICAL RIB AT STA. X=35
INNER WING-CONSTANT SECTION



OUTER WING-NORMAL RIBS

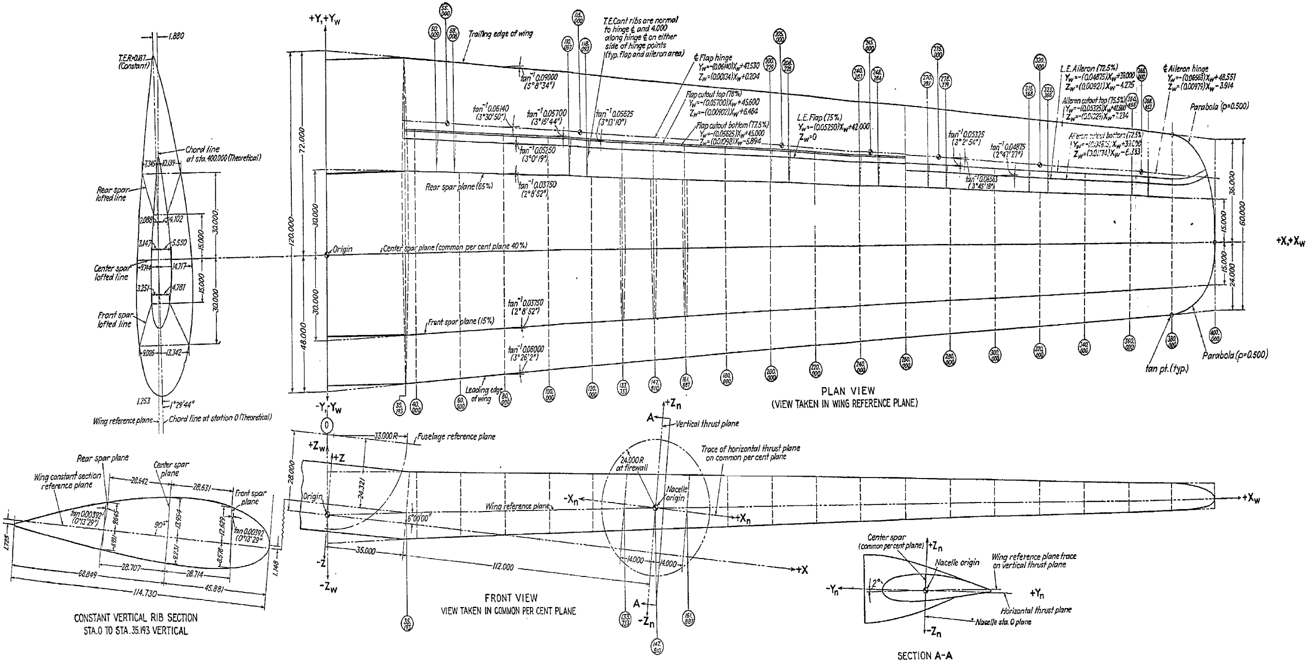


Fig. 9.10.—Master diagram.

The true angle between the two planes is equal to the true angle between these two normals. It is given by

$$\cos A = \sin \phi \sin \theta.$$

This example is typical. The charts in Figs. 9.7 and 9.8 show the cosines of the true angles between these fundamental planes.

In calculating true angles between lines and planes and true angles between two planes, it is necessary to have the direction ratios of normals to the planes. Certain basic planes occur so often in these calculations that it is convenient to tabulate the direction ratios of normals to them. The tables shown on page 180 are based on Figs. 3.5, 3.10, and 3.12, and also apply to the master diagram (Fig. 9.10).

Example 2. Find the true angle between the front spar plane and a water line plane, in the case of a wing rigged by the chord plane system.

From the chart on the chord plane system, the direction ratios of normals to these two planes in the wing chord plane system of axes are

$$\sin \beta : -\cos \beta : 0,$$
$$\sin \phi \cos \theta : -\sin \theta : \cos \phi \cos \theta.$$

The true angle between these two normals is therefore given by

$$\cos A = \sin \beta \sin \phi \cos \theta + \cos \beta \sin \theta.$$

9.5. Equations of basic wing lines in x_w, y_w, z_w coordinates. One of the first basic mathematical jobs that must be done on a wing is to derive the equations of the wing per cent lines, hinge center lines, aileron and flap cutout lines, etc. Table 1 shows a typical tabulation of the equations of these lines in their lofted position, that is, with reference to the x_w, y_w, z_w system of axes.

Consider, as an example, the 15 per cent top lofted line. In Table 1, the coordinates of this line for wing station $x_w = 0$ are (0, -30.000, 13.342). The coordinates of the 15 per cent top lofted line for wing station $x_w = 400$ are (400, -15.000, 4.781). These coordinates are read directly from Table 1. Calculate the direction ratios of the line from these two sets of coordinates. The direction ratios are

$$1 : 0.03750 : -0.02140.$$

The equations of the line, as derived from the two sets of coordinates and the direction ratios, are

$$y_w = 0.03750x_w - 30.000,$$
$$z_w = -0.02140x_w + 13.342.$$

These two equations need not be derived in this way. Once Table I has been prepared, the equations of the per cent lines

TABLE I

Per cent	Station $x_w = 0$			Station $x_w = 400$			Tangents		
	y_w	z_w (top)	z_w (bot.)	y_w	z_w (top)	z_w (bot.)	y_w	z_w (top)	z_w (bot.)
0	−48.000	1.253	1.253	−24.000	0	0	0.06000	−0.00313	−0.00313
0.1	−47.880	2.177	− 0.018	−23.940	0.493	−0.336	0.05985	−0.00421	−0.00080
0.25	−47.700	2.854	− 0.601	−23.850	0.774	−0.545	0.05963	−0.00520	0.00014
0.5	−47.400	3.688	− 1.276	−23.700	1.084	−0.788	0.05925	−0.00651	0.00122
1.0	−46.800	4.793	− 2.240	−23.400	1.476	−1.110	0.05850	−0.00829	0.00282
1.75	−45.900	5.972	− 3.290	−22.950	1.897	−1.454	0.05738	−0.01019	0.00459
2.5	−45.000	6.863	− 4.062	−22.500	2.225	−1.716	0.05625	−0.01160	0.00586
3.75	−43.500	8.047	− 5.070	−21.750	2.678	−2.046	0.05438	−0.01342	0.00756
5	−42.000	9.012	− 5.856	−21.000	3.043	−2.302	0.05250	−0.01492	0.00888
7.5	−39.000	10.524	− 7.044	−19.500	3.637	−2.681	0.04875	−0.01722	0.01091
10	−36.000	11.700	− 7.910	−18.000	4.100	−2.942	0.04500	−0.01900	0.01242
15	−30.000	13.342	− 9.026	−15.000	4.781	−3.251	0.03750	−0.02140	0.01444
20	−24.000	14.358	− 9.676	−12.000	5.222	−3.392	0.03000	−0.02284	0.01571
25	−18.000	14.960	− 9.954	− 9.000	5.503	−3.420	0.02250	−0.02364	0.01633
30	−12.000	15.174	−10.021	− 6.000	5.628	−3.370	0.01500	−0.02386	0.01663
35	− 6.000	15.074	− 9.938	− 3.000	5.631	−3.275	0.00750	−0.02361	0.01666
40	0	14.717	− 9.744	0	5.550	−3.147	0	−0.02292	0.01649
45	6.000	14.131	− 9.430	3.000	5.385	−2.989	−0.00750	−0.02186	0.01610
50	12.000	13.354	− 8.996	6.000	5.144	−2.798	−0.01500	−0.02052	0.01549
55	18.000	12.430	− 8.497	9.000	4.843	−2.578	−0.02250	−0.01897	0.01480
60	24.000	11.354	− 7.942	12.000	4.500	−2.339	−0.03000	−0.01714	0.01401
65	30.000	10.139	− 7.354	15.000	4.102	−2.088	−0.03750	−0.01509	0.01314
70	36.000	8.810	− 6.708	18.000	3.661	−1.828	−0.04500	−0.01287	0.01220
75	42.000	7.373	− 6.042	21.000	3.170	−1.566	−0.05250	−0.01051	0.01119
80	48.000	5.825	− 5.323	24.000	2.646	−1.289	−0.06000	−0.00795	0.01009
85	54.000	4.153	− 4.564	27.000	2.080	−0.997	−0.06750	−0.00518	0.00892
90	60.000	2.364	− 3.791	30.000	1.469	−0.702	−0.07500	−0.00224	0.00772
95	66.000	0.416	− 2.975	33.000	0.830	−0.430	−0.08250	0.00104	0.00636
100	72.000	−1.880	− 1.880	36.000	0	0	−0.09000	0.00470	0.00470
L.E. flap	42.000	0	....	28.350*	0*	....	−0.05250	0	....
L.E. aileron	39.000	....	− 4.275	19.500	......	−0.591	−0.04875	....	0.00921
℄ aileron hinge	48.551	....	− 3.914	22.299	......	0.002	−0.06563	....	0.00979
℄ flap hinge	47.530	0.204	....	31.566*	0.552*	....	−0.06140	0.00134	....
Flap cutout	45.600	6.464	....	30.780*	4.119*	....	−0.05700	−0.00902	....
Flap cutout	45.000	....	− 5.894	30.375*	......	−3.039*	−0.05625	....	0.01098
Aileron cutout	42.600	7.234	....	21.300	3.118	....	−0.05325	−0.01029	....
Aileron cutout	39.000	...	− 6.383	19.500	......	−1.687	−0.04875	....	0.01174

* At station $x_w = 260$.

can be read directly from the table. For example, in the equation

$$y_w = 0.03750x_w - 30.000$$

the number 0.03750 can be found directly in the table in the

vertical column marked y_w under Tangents, and on the 15 per cent horizontal row. The number -30.000 can be found directly in the vertical column marked y_w under Station $x_w = 0$, and on the 15 per cent horizontal row. Similarly, in the equation

$$z_w = -0.02140x_w + 13.342$$

the number -0.02140 can be found in the table in the vertical column marked z_w (top) under Tangents, and on the 15 per cent horizontal row. The number 13.342 can be found in the vertical column marked z_w (top) under Station $x_w = 0$, and on the 15 per cent horizontal row.

In like manner, the equations of all the basic wing lines can be read directly from Table I. In order to understand and appreciate the table, it is advisable to derive some of the equations, as explained previously, and then to check the answers by comparing them with the results as read directly from the table.

Example. Using the information in Table I as given, it is possible to write the equations of any of the given lines by reading the table directly. No calculations are necessary. For instance, the equations of the 75 per cent bottom lofted line are

$$\begin{aligned} y_w &= -0.05250x_w + 42.000, \\ z_w &= 0.01119x_w - 6.042. \end{aligned}$$

Exercises

Determine the equations of the following lines, using Table I and making no calculations:

1. 20 per cent top lofted line.
2. 35 per cent bottom lofted line.
3. 45 per cent bottom lofted line.
4. Leading edge of the aileron.
5. Leading edge of the flap.

Table I refers to the wing designed on the master diagram and loft layout (Figs. 9.9 and 9.10).

9.6. Calculation of normal wing ribs. The next problem that confronts the basic calculations group is to furnish the coordinates of the lines listed in Table I at any given rib station, so that the engineering department can use these normal rib coordinates for basic layout work before the loft has completed the loft layout of the wing.

In order to calculate the y_w coordinates and z_w coordinates for a normal rib at a given wing station, it is merely necessary to substitute the x_w value indicating the wing station in the equations for y_w and z_w in Table I. For example, the shape of the normal rib at wing station $x_w = 300$ can be completely determined by substituting 300 for x_w in the equations of the various per cent lines. The resulting y_w and z_w values determine the offsets to the contour of the normal rib in the body plan (end) view of the rib. The y_w and z_w values can be plotted, and the result is the shape of the normal rib.

Example. Find the coordinates of the point of intersection of the 20 per cent top lofted line with the wing rib station $x_w = 400$.

The equations of the 20 per cent top lofted line are

$$\begin{aligned} y_w &= \quad 0.03000x_w - 24.000, \\ z_w &= -0.02284x_w + 14.358. \end{aligned}$$

The equation of the normal rib station plane is $x_w = 400$. Substitute 400 for x_w in the equations of the line. The results are $y_w = -12.000$ and $z_w = 5.222$. Therefore the coordinates of the required point are (400, -12.000, 5.222). It should be noted that this particular station is tabulated in Table I. Any intermediate normal rib can be calculated by this same method.

Exercises

1. Determine the coordinates of the basic wing lines as given in Table I for rib station $x_w = 400$ by using the coordinates of the points at station $x_w = 0$ and the direction ratios of the lines.

2. Determine the coordinates of the points of intersection of the basic per cent lines in Table I and wing station $x_w = 200$ and plot the points, thus determining the contour of the normal rib at that station.

9.7. Calculation of vertical wing ribs. In order to determine the shape of a vertical wing rib, say station $x = 35$, the procedure described in Art. 9.6 can be followed. Notice that $x = 35$ is a plane parallel to the plane of symmetry, since the x without the subscript w refers to the rigged system of axes. The equations of the basic wing lines as given in Table I are stated in terms of the x_w, y_w, z_w system of axes. If there are many vertical ribs to be calculated, it is more expedient to determine the equations of the basic wing lines with reference to the x, y, z system of axes (rigged position), and then substitute the x value of the vertical rib plane in the equations of the basic wing lines to

determine the coordinates of the lofted line (outside shape) of the vertical rib plane.

Example 1. Using the information as given in Table I, determine the equations of the 10 per cent top lofted line in rigged position, *i.e.*, with respect to the x, y, z system of axes.

First, obtain the coordinates of a point on this line and the direction ratios of the line, directly from Table I.

$$x_w = 0, \qquad y_w = -36.000, \qquad z_w = 11.700.$$
$$1:0.04500:-0.01900.$$

TABLE II

Per cent	Station $x = 0$				Tangents			
	Top L.L.		Bottom L.L.		Top L.L.		Bottom L.L.	
	y	z	y	z	y	z	y	z
0	−47.992	1.259	−47.992	1.259	0.06031	0.10194	0.06031	0.10194
0.1	−47.866	2.188	−47.880	− 0.018	0.06015	0.10085	0.06017	0.10430
0.25	−47.682	2.868	−47.704	− 0.604	0.05993	0.09985	0.05996	0.10525
0.5	−47.377	3.706	−47.408	− 1.283	0.05954	0.09853	0.05958	0.10634
1.0	−46.771	4.815	−46.814	− 2.253	0.05877	0.09673	0.05884	0.10796
1.75	−45.864	5.998	−45.920	− 3.310	0.05763	0.09481	0.05772	0.10975
2.5	−44.959	6.892	−45.024	− 4.087	0.05649	0.09339	0.05659	0.11103
3.75	−43.454	8.080	−43.529	− 5.102	0.05460	0.09156	0.05472	0.11275
5	−41.950	9.047	−42.032	− 5.894	0.05271	0.09004	0.05284	0.11409
7.5	−38.946	10.563	−39.036	− 7.091	0.04893	0.08773	0.04907	0.11615
10	−35.945	11.741	−36.037	− 7.964	0.04516	0.08593	0.04531	0.11768
15	−29.948	13.385	−30.036	− 9.090	0.03762	0.08352	0.03776	0.11973
20	−23.955	14.402	−24.031	− 9.745	0.03009	0.08207	0.03022	0.12101
25	−17.965	15.005	−18.024	−10.026	0.02257	0.08126	0.02266	0.12164
30	−11.976	15.219	−12.016	−10.094	0.01504	0.08104	0.01511	0.12195
35	− 5.988	15.119	− 6.008	−10.010	0.00752	0.08129	0.00755	0.12198
40	0	14.763	0	− 9.815	0	0.08199	0	0.12181
45	5.989	14.176	6.007	− 9.498	−0.00752	0.08305	−0.00755	0.12141
50	11.979	13.399	12.014	− 9.060	−0.01505	0.08440	−0.01511	0.12079
55	17.971	12.474	18.020	− 8.557	−0.02258	0.08596	−0.02266	0.12009
60	23.964	11.396	24.025	− 7.998	−0.03011	0.08781	−0.03021	0.11929
65	29.960	10.179	30.029	− 7.396	−0.03765	0.08987	−0.03776	0.11841
70	35.958	8.847	36.032	− 6.754	−0.04519	0.09211	−0.04531	0.11746
75	41.959	7.405	42.033	− 6.082	−0.05273	0.09449	−0.05285	0.11643
80	47.963	5.852	48.034	− 5.358	−0.06028	0.09707	−0.06039	0.11532
85	53.971	4.174	54.032	− 4.593	−0.06783	0.09987	−0.06794	0.11413
90	59.981	2.376	60.030	− 3.815	−0.07540	0.10284	−0.07547	0.11292
95	65.996	0.418	66.026	− 2.993	−0.08296	0.10616	−0.08301	0.11154
100	72.018	−1.891	72.018	− 1.891	−0.09054	0.10986	−0.09054	0.10986
L.E. flap	42.000	0	....	....	−0.05279	0.10511	....	....
℄ flap hinge	47.529	0.205	....	....	−0.06175	0.10646	....	....
Flap cutout	45.561	6.493	....	....	−0.05726	0.09600	....	....
Flap cutout	....	....	45.035	− 5.933	....	....	−0.05663	0.11622

TABLE III

Per cent	Station $x = 35$					
	Top L.L.			Bottom L.L.		
	y	z (station 0 ref.)	z (station 35 ref.)	y	z (station 0 ref.)	z (station 35 ref.)
0	−45.881	4.827	1.148	−45.881	4.827	1.148
0.1	−45.761	5.718	2.039	−45.774	3.633	−0.046
0.25	−45.584	6.363	2.684	−45.605	3.080	−0.599
0.5	−45.293	7.155	3.476	−45.323	2.439	−1.240
1.0	−44.714	8.201	4.522	−44.755	1.526	−2.153
1.75	−43.847	9.316	5.637	−43.900	0.531	−3.148
2.5	−42.982	10.161	6.482	−43.043	−0.201	−3.880
3.75	−41.543	11.285	7.606	−41.614	−1.156	−4.835
5	−40.105	12.198	8.519	−40.183	−1.901	−5.580
7.5	−37.233	13.634	9.955	−37.319	−3.026	−6.705
10	−34.364	14.749	11.070	−34.451	−3.845	−7.524
15	−28.631	16.308	12.629	−28.714	−4.899	−8.578
20	−22.902	17.274	13.595	−22.973	−5.510	−9.189
25	−17.175	17.849	14.170	−17.231	−5.769	−9.448
30	−11.450	18.055	14.376	−11.487	−5.826	−9.505
35	− 5.725	17.964	14.285	− 5.744	−5.741	−9.420
40	0	17.633	13.954	0	−5.552	−9.231
45	5.726	17.083	13.404	5.743	−5.249	−8.928
50	11.452	16.353	12.674	11.485	−4.832	−8.511
55	17.181	15.483	11.804	17.227	−4.354	−8.033
60	22.910	14.469	10.790	22.968	−3.823	−7.502
65	28.642	13.324	9.645	28.707	−3.252	−6.931
70	34.376	12.071	8.392	34.446	−2.643	−6.322
75	40.113	10.712	7.033	40.183	−2.007	−5.686
80	45.853	9.249	5.570	45.920	−1.322	−5.001
85	51.597	7.669	3.990	51.654	−0.598	−4.277
90	57.342	5.975	2.296	57.389	0.137	−3.542
95	63.092	4.134	0.455	63.121	0.911	−2.768
100	68.849	1.954	−1.725	68.849	1.954	−1.725
L.E. flap......	40.152	3.679	0	....	....	....
℄ flap hinge..	45.368	3.931	0.252	....	....	....
Flap cutout...	43.557	9.853	6.174	....	....	....
Flap cutout...	....	......	......	43.053	−1.865	−5.544

The rotation of axes equations are

$$\begin{aligned} x &= x_w \cos \phi - z_w \sin \phi, \\ y &= y_w, \\ z &= x_w \sin \phi + z_w \cos \phi. \end{aligned}$$

The angle of dihedral for this wing is $\phi = 6°$. See the wing master diagram (Fig. 9.10). The new coordinates of the point are

$$x = -1.223, \qquad y = -36.000, \qquad z = 11.636.$$

The direction ratios of the line in the x, y, z system are

$$0.99651:0.04500:0.08563 \qquad \text{or} \qquad 1:0.04516:0.08593.$$

Using this point and these direction ratios to write the equations of the line,

$$\begin{aligned} y &= 0.04516x - 35.945, \\ z &= 0.08593x + 11.741. \end{aligned}$$

The equations of all the basic wing lines in the x, y, z system are tabulated in Table II. These equations were derived by the method described in Example 1.

Example 2. Using Table II directly, with no calculations necessary, the equations of the 50 per cent bottom lofted line are

$$\begin{aligned} y &= -0.01511x + 12.014, \\ z &= 0.12079x - 9.060. \end{aligned}$$

To determine the coordinates of the lofted line of vertical rib station $x = 35$, substitute $x = 35$ in the equations of the basic wing lines tabulated in Table II. The complete set of answers are shown in Table III.

The z values of the points are given with relation to the wing reference plane trace at station $x = 0$ (plane of symmetry of the airplane) and the wing reference plane trace at station $x = 35$. The values in Table III can be used to lay out the shape of the vertical rib at station $x = 35$.

Exercises

1. Using Table I, derive the equations for the following basic wing lines in rigged position. Use the rotation of axes equations. Check the answers obtained by this method with the values as tabulated in Table II.

(*a*) 10 per cent top lofted line (L.L.).
(*b*) 25 per cent bottom lofted line.
(*c*) Leading edge (L.E.) of the flap.
(*d*) 40 per cent top lofted line.
(*e*) 75 per cent top lofted line.
(*f*) 95 per cent bottom lofted line.
(*g*) 35 per cent top lofted line.

2. Using the equations of the basic wing lines as tabulated in Table II, determine the coordinates of the lofted line of vertical rib station $x = 35$

at the following points. Check the answers obtained by this method with the values as tabulated in Table III.

(*a*) 10 per cent top lofted line.
(*b*) 25 per cent bottom lofted line.
(*c*) Leading edge of the flap.
(*d*) 40 per cent top lofted line.
(*e*) 75 per cent top lofted line.
(*f*) 95 per cent bottom lofted line.
(*g*) 35 per cent top lofted line.

9.8. Wing bevels (flange angles). The term *bevel*, as it is used in the aircraft industry, is synonymous with the word *angle.* The bevels on a normal wing rib are the angles that are made by the plane of a normal wing rib with the outer covering (skin) of the airplane. Since the outer skin of the wing is a curved surface, the bevel of a normal wing rib varies along its lofted line from point to point. For obtaining a very close approximation of the bevel of the flange of the wing rib, which attaches the skin to the rib, the skin is taken as a plane at the point where the bevel is read (see the loft layout, Fig. 9.9).

The loft layout shows how the bevel is read at the 90 per cent top lofted line of the vertical rib station $x = 35$, measuring outboard. Here Δy and Δz are found in Table II under Tangents of Top L.L. of y and z, respectively. Tangent $A = 0.075$ is measured on a full-sized layout of a wing, as illustrated on the loft layout (Fig. 9.9). The angle is an open angle. $A = 4°17'$. Therefore the true bevel is open $4°17'$, or the true angle between the wing skin and the plane of the vertical rib at station $x = 35$ is equal to $94°17'$. The angle in practical use for most cases has an allowable tolerance of at least $\pm 10'$. For this reason, bevels are usually picked up from the loft layout with the help of some analytic geometry, as shown on the loft layout.

To obtain the bevels on a normal rib, follow the same procedure as described in the last paragraph for a vertical rib, but use Table I. It will be found that these points lie on the per cent lines as shown in the loft layout. Since this is true, it is simple to plot these points by using proportional distances along the per cent lines. Points are plotted on the loft layout for the stations at 20-in. intervals. Therefore $\frac{1}{20}$ of that distance would be the proper distance for determining the points on the lofted line of a normal rib 1 in. away. This will be adequate for determining the bevels of the normal ribs. These values

are usually calculated at the same time as the equations of the basic lines, since it is advisable to make the loft layout of the wing using these factors.

For example, the direction ratios of the 15 per cent top lofted line from Table I are

$$1:0.03750:-0.02140.$$

The factor is therefore

$$\sqrt{(0.03750)^2 + (-0.02140)^2}.$$

On some wings it is possible to use identical bevels on all parallel ribs at the same per cent line.

Exercises

Determine the following bevels, all bevels being measured outboard.

1. Normal rib at station $x_w = 300$.

(*a*) 15 per cent top lofted line.
(*b*) 25 per cent top lofted line.
(*c*) 35 per cent bottom lofted line.
(*d*) 50 per cent bottom lofted line.
(*e*) 75 per cent bottom lofted line.

2. Vertical rib at station $x = 35$.

(*a*) 15 per cent top lofted line.
(*b*) 25 per cent top lofted line.
(*c*) 35 per cent bottom lofted line.
(*d*) 50 per cent bottom lofted line.
(*e*) 75 per cent bottom lofted line.

9.9. Angle made on a vertical rib plane by the intersections of the front spar plane and the wing reference plane. Consider the problem of determining the angle made on the vertical rib plane at station $x = 35$ by the intersections of the front spar plane and the wing reference plane. Is the aft lower angle greater or smaller than 90°? See the master diagram (Fig. 9.10).

The direction ratios of a normal to the front spar plane are

$$-\tan\beta:1:0.$$

The direction ratios of a normal to the vertical rib plane are

$$1:0:-\tan\phi.$$

Both these sets of direction ratios are with respect to the x_w, y_w, z_w system of axes. The direction ratios of the line of intersection

of the front spar plane and the vertical rib plane are obtained by cross-multiplying

$$-\tan\beta : 1 : 0$$
$$1 : 0 : -\tan\phi.$$

The results are

$$-\tan\phi : -\tan\beta\tan\phi : -1.$$

The direction ratios of the line of intersection of the wing reference plane and the vertical rib plane are

$$0:1:0.$$

The required angle is given by

$$\cos A = \frac{-\tan\beta\tan\phi}{\sqrt{\tan^2\phi + \tan^2\beta\tan^2\phi + 1}}.$$
$$\cos A = \frac{-\tan\beta\tan\phi}{\sqrt{\sec^2\phi + \tan^2\beta\tan^2\phi}}.$$
$$\tan A = \frac{-\sec\phi}{\tan\beta\tan\phi}.$$
$$\tan A = \frac{-1}{\tan\beta\sin\phi}.$$
$$\cot A = -\tan\beta\sin\phi.$$
$$\cot A = -(0.03750)(0.10453).$$
$$\cot A = -0.00392.$$
$$A = 90°13'29''.$$

Exercise

Find the angle made on the vertical rib plane at station $x = 35$ by the intersections of the rear spar plane and the wing reference plane (see the master diagram, Fig. 9.10). *Ans.* 89°46′31″.

9.10. True angle between a vertical rib plane and the front spar plane. Consider the problem of determining the true angle between a vertical rib plane and the front spar plane (see the master diagram, Fig. 9.10).

The direction ratios of a normal to the vertical rib plane in x_w, y_w, z_w position are

$$1:0:-\tan \qquad \text{or} \qquad 1:0:-0.10510.$$

The direction ratios of a normal to the front spar plane in x_w, y_w, z_w position are

$$-0.03750:1:0.$$

The required angle is given by

$$\cos A = \frac{-0.03750}{\sqrt{1 + (-0.10510)^2}\sqrt{(-0.03750)^2 + 1}}.$$
$$\cos A = -0.03727.$$
$$A = 92°8'8''.$$

Exercise

Find the true angle between the rear spar plane and a vertical rib plane (see the master diagram, Fig. 9.10). *Ans.* 87°51′52″.

9.11. True distance along the flap hinge center line between hinge points, as measured from the first hinge point. Consider the problem of determining the true distance along the flap hinge center line between hinge points, as measured from the first hinge point at station $x_w = 55.000$ (see the master diagram, Fig. 9.10).

The direction ratios of the flap hinge center line are

$$1 : -0.06140 : 0.00134.$$

These direction ratios may be obtained from the equations of this line on the master diagram. The length for one unit change in x_w value is

$$\sqrt{1 + (-0.06140)^2 + (0.00134)^2},$$

or 1.00188. Therefore the required distances are given by

$$d = (1.00188)(x_w - 55).$$

The distances for the various values of x_w at the hinge points on the center line of flap hinge are

$$x_w = 55.000. \qquad d = 0.$$
$$x_w = 115.000. \qquad d = 60.113.$$
$$x_w = 205.000. \qquad d = 150.282.$$
$$x_w = 245.000. \qquad d = 190.357.$$

Exercise

Find the true distance along the aileron hinge center line between hinge points as measured from the first hinge point at station $x_w = 275.000$ (see the master diagram, Fig. 9.10). *Ans.* $x_w = 275.000$. $d = 0$.
$x_w = 320.000$. $d = 45.099$.
$x_w = 365.000$. $d = 90.198$.

9.12. True angle between the plane of the flap hinge bracket and the rear spar plane. The flap hinge bracket is taken to be a

plane normal to the flap hinge center line. Consider the problem of determining the true angle between this plane and the rear spar plane (see the master diagram, Fig. 9.10).

The direction ratios of a normal to the rear spar plane in x_w, y_w, z_w position are

$$0.03750:1:0.$$

The direction ratios of a normal to the plane of the bracket are the direction ratios of the flap hinge center line. They are

$$1:-0.06140:0.00134.$$

The required angle is given by

$$\cos A = \frac{0.03750 - 0.06140}{\sqrt{1 + (-0.03750)^2}\sqrt{1 + (-0.06140)^2 + (0.00134)^2}}.$$

$$\cos A = -0.02384.$$

$$A = 91°21'58''.$$

Exercise

Find the true angle between the planes of the aileron hinge brackets (planes normal to the aileron hinge center line) and the rear spar plane (see the master diagram, Fig. 9.10). *Ans.* $A = 91°36'27''$.

9.13. Rotation of axes equations for flap position to wing lofted position. Consider the problem of determining the rotation of axes equations for converting coördinates from the flap position (x_f, y_f, z_f) to the wing lofted position (x_w, y_w, z_w). See the master diagram (Fig. 9.10).

The x_f axis is the flap hinge center line. The direction ratios of the x_f axis are therefore

$$1:-0.06140:0.00134.$$

The y_f axis is perpendicular to the x_f axis and also perpendicular to the z_w axis. This position of the y_f axis is selected in order that the trace of the $x_f y_f$ plane and the trace of the $x_w y_w$ plane on a plane normal to the flap hinge center line will be parallel.

The direction ratios of the y_f axis may therefore be obtained by cross-multiplying

$$1:-0.06140:0.00134$$
$$0:0:1.$$

The results are

$$-0.06140:-1:0 \qquad \text{or} \qquad 0.06140:1:0.$$

The z_f axis is normal to the x_f axis and the y_f axis. Its direction ratios may be obtained by cross-multiplying the direction ratios of the x_f axis and y_f axis.

The box for the rotation of axes equations is as follows:

	x_w	y_w	z_w
x_f	0.9981194	−0.0612845	0.0013375
y_f	0.0612846	0.9981203	0
z_f	−0.0013350	0.0000820	0.9999990

The rotation of axes equations from flap position to wing lofted position are

$$\begin{aligned} x_w &= 0.9981194x_f + 0.0612846y_f - 0.0013350z_f, \\ y_w &= -0.0612845x_f + 0.9981203y_f + 0.0000820z_f, \\ z_w &= 0.0013375x_f + 0.9999990z_f. \end{aligned}$$

The rotation of axes equations from wing lofted position to flap position are

$$\begin{aligned} x_f &= 0.9981194x_w - 0.0612845y_w + 0.0013375z_w, \\ y_f &= 0.0612846x_w + 0.9981203y_w, \\ z_f &= -0.0013350x_w + 0.0000820y_w + 0.9999990z_w. \end{aligned}$$

Notice that seven-place decimals are used here. In a case like this, when the flap hinge center line makes such a small angle with the wing reference plane, five-place decimals are not adequate.

Consider the matter of determining the origins of these two systems of axes with respect to each other (see the master diagram, Fig. 9.10). The origin of the flap position is

$$\begin{aligned} x_f &= 0. & x_w &= 0. \\ y_f &= 0. & y_w &= 47.530. \\ z_f &= 0. & z_w &= 0.204. \end{aligned}$$

The origin of the wing lofted position is

$$\begin{aligned} x_w &= 0. & x_f &= 2.913. \\ y_w &= 0. & y_f &= -47.441. \\ z_w &= 0. & z_f &= -0.208. \end{aligned}$$

Exercise.

Determine the rotation of axes equations for aileron position (x_a, y_a, z_a) to wing lofted position (x_w, y_w, z_w), and the coordinates of the origin of each system with respect to the other system (see the master diagram, Fig. 9.10).

Ans.

	x_w	y_w	z_w
x_a	0.9978056	−0.0654860	0.0097685
y_a	0.0654891	0.9978533	0
z_a	−0.0097475	0.0006397	0.9999522

Aileron origin

$$x_a = 0. \quad x_w = 0.$$
$$y_a = 0. \quad y_w = 48.551.$$
$$z_a = 0. \quad z_w = -3.914.$$

Wing origin

$$x_w = 0. \quad x_a = 3.218.$$
$$y_w = 0. \quad y_a = -48.447.$$
$$z_w = 0. \quad z_a = 3.883.$$

9.14. Intersection of a line and a plane. Consider the problem of determining the x_w coordinate of the point of intersection of the canted trailing edge structure in the flap area with the wing reference plane trace on the rear spar (see the master diagram, Fig. 9.10).

The coordinates of the first hinge point are

$$x_w = 55.000,$$
$$y_w = 44.153,$$
$$z_w = 0.278.$$

The equation of the plane perpendicular to the flap hinge center line at station $x_w = 55$ is

$$(0.9981194)(x_w - 55.000) - (0.0612845)(y_w - 44.153) + (0.0013375)(z_w - 0.278) = 0.$$
$$0.9981194x_w - 0.0612845y_w + 0.0013375z_w = 52.191.$$

For a more general solution, where more than one problem of this type is to be solved, let $52.191 = d$. Here d is the distance from the origin to the planes perpendicular to the flap hinge center line at the hinge points. Then the equation of the plane is

$$0.9981194x_w - 0.0612845y_w + 0.0013375z_w = d.$$

The trace of the wing reference plane on the rear spar is a line, whose equations are

$$y_w = -0.03750x_w + 30.000,$$
$$z_w = 0.$$

Solving for x_w in terms of d,

$$x_w = 0.9995826d + 1.838.$$

To find the x_w station at the rear spar-wing reference plane trace for the ribs 4 in. on either side of the flap hinge center line at station $x_w = 55$ (see the master diagram):

$$x_w = 0.9995826(d \pm 4) + 1.838.$$

The inboard station is

$$x_w = 50.009.$$

The outboard station is

$$x_w = 58.006.$$

To find the d values for all other cant stations, in order to solve for the x_w values at the rear spar-wing reference plane trace, use

$$d = 52.191 + (1.0018841)(x_w - 55) \pm 4.$$

For the second hinge point at station $x_w = 115.000$, where $d = 112.304$, the values of d and x_w are

$$d = 108.304, \qquad x_w = 110.097.$$
$$d = 116.304, \qquad x_w = 118.093.$$

For the third hinge point at station $x_w = 205.000$, where $d = 202.474$, the values of d and x_w are

$$d = 198.474, \qquad x_w = 200.229.$$
$$d = 206.474, \qquad x_w = 208.226.$$

For the hinge point at station $x_w = 245.000$, where $d = 242.549$, the values of d and x_w are

$$d = 238.549, \qquad x_w = 240.287.$$
$$d = 246.549, \qquad x_w = 248.284.$$

Exercise

Find the x_w coordinates of the points of intersection of the canted trailing edge structure in the aileron area with the wing reference plane trace on the rear spar (see the master diagram, Fig. 9.10).

Ans. $x_w = 270.281.$
$x_w = 278.279.$
$x_w = 315.368.$
$x_w = 323.366.$
$x_w = 360.456.$
$x_w = 368.453.$

9.15. Rotation of axes equations for nacelle position to wing lofted position. See the master diagram (Fig. 9.10) and Art. 8.10. The box for the rotation of axes equations for nacelle position (x_n, y_n, z_n) to wing lofted position (x_w, y_w, z_w) is as follows:

	x_w	y_w	z_w
x_n	0.99452	0	−0.10453
y_n	−0.00365	0.99939	−0.03471
z_n	0.10447	0.03490	0.99391

The nacelle origin is at

$$\begin{aligned} x_n &= 0. & x_w &= 147.810. \\ y_n &= 0. & y_w &= 0. \\ z_n &= 0. & z_w &= 0. \end{aligned}$$

The wing origin is at

$$\begin{aligned} x_w &= 0. & x_n &= -147.000. \\ y_w &= 0. & y_n &= 0.540. \\ z_w &= 0. & z_n &= -15.442. \end{aligned}$$

As an example in the use of these equations, determine the equation of the front spar plane in nacelle position (see the master diagram).

A point on the front spar plane is

$$\begin{aligned} x_w &= 0. & x_n &= -147. \\ y_w &= -30. & y_n &= -29.4417. \\ z_w &= 0. & z_n &= -16.4890. \end{aligned}$$

The direction ratios of a normal to the front spar plane are

$$x_w = -0.03750. \qquad x_n = -0.03731.$$
$$y_w = 1. \qquad y_n = 1.$$
$$z_w = 0. \qquad z_n = 0.03100.$$

The equation of the front spar plane in nacelle position is

$$(-0.03731)(x_n + 147) + (1)(y_n + 29.4417) + (0.03100)(z_n + 16.4890) = 0.$$
$$-0.03731x_n + y_n + 0.03100z_n = -24.468.$$

Exercise

Determine the equation of the rear spar plane in nacelle position (see the master diagram, Fig. 9.10).

Ans. $0.03732x_n + y_n + 0.03885z_n = 24.476.$

9.16. Angle between a line and a plane. The center line of the main landing gear bearing is parallel to the x axis. Determine the true angle between the center line of the main landing gear bearing and the front spar plane, to which the fitting that locates the landing gear bearings is attached.

The direction ratios of the center line of the bearing in x, y, z position are $1:0:0$. In x_w, y_w, z_w position they are $1:0:-\tan\phi$.

The direction ratios of a normal to the front spar plane in x_w, y_w, z_w position are $-\tan\beta:1:0$.

The true angle is given by

$$\cos A = \frac{-0.03750}{\sqrt{1 + (-0.10510)^2}\,\sqrt{1 + (-0.03750)^2}}$$
$$\cos A = -0.03727.$$

Since we are finding the angle between a line and a plane, we have

$$A = 2°8'8''.$$

Compare this problem with the problem in Art. 9.10. The two problems are equivalent. The equivalence of problems is worth noting and watching for since it occurs so often. Problems that are stated differently are often equivalent problems.

Exercise

Determine the angle made on the front spar plane by the projection of the center line of bearing on the front spar plane and the wing reference plane trace on the front spar plane. *Ans.* $6°0'14''$.

9.17. Remarks on accuracy. By its very nature, accuracy in measurements is a relative rather than an absolute concept.

The accuracy of the calculations is based upon the accuracy of the available information and data, and these in turn depend upon the uses to which they are to be put, and upon certain physical limitations. Slide-rule accuracy may be warranted under some circumstances, but it is usually valuable only as a rough check. In this book we have used five-decimal-place accuracy for convenience. Usually the calculations necessary in work of this type are performed on calculating machines. Seven-decimal-place accuracy can be justified in certain instances. It is convenient to use a calculating machine to the limit of its keyboard. A careful discussion of significant figures in sequences of operations as varied and as numerous as those employed here would be possible, but of doubtful practical value. No general rules can be laid down. The matter of accuracy is worth careful consideration, and the final decisions must be based upon the requirements of the job to be done.

9.18. General remarks on the master diagram, loft layout, and basic data sheets. The master diagram is a source of many problems which illustrate the use of solid analytic geometry as applied to the airplane. Used in conjunction with the loft layout and the basic data sheets, it can be used to illustrate all the principles discussed in the previous chapters. A short list of such applications is

1. Calculation of contours of all normal ribs.
2. Calculation of contours of all vertical ribs.
3. Distance between hinge points.
4. Distance along leading edge, trailing edge, spar lofted lines, per cent lines, etc., between normal ribs and between vertical ribs.
5. True angles between basic planes.
6. Equations of basic lines and planes.
7. Angle between a normal rib trace and a vertical rib trace on a spar plane, and other similar combinations of basic planes.
8. Distance from a hinge point to the top and bottom lofted lines of the rear spar, as measured in the plane of the hinge bracket (which is normal to the hinge center line).
9. Rotation of axes from wing lofted position to various subassembly positions, such as trailing edge section, center section, nose section, and finally final assembly position (rigged position).

CHAPTER 10

CONIC SECTIONS. GRAPHICAL TREATMENT

The intersection of a plane and a right circular cone is a conic section. The cone is composed of two parts, or nappes (see Fig. 10.1). If the plane does not pass through the vertex of the cone, the section is an ellipse, parabola, or hyperbola (see Fig. 10.1).

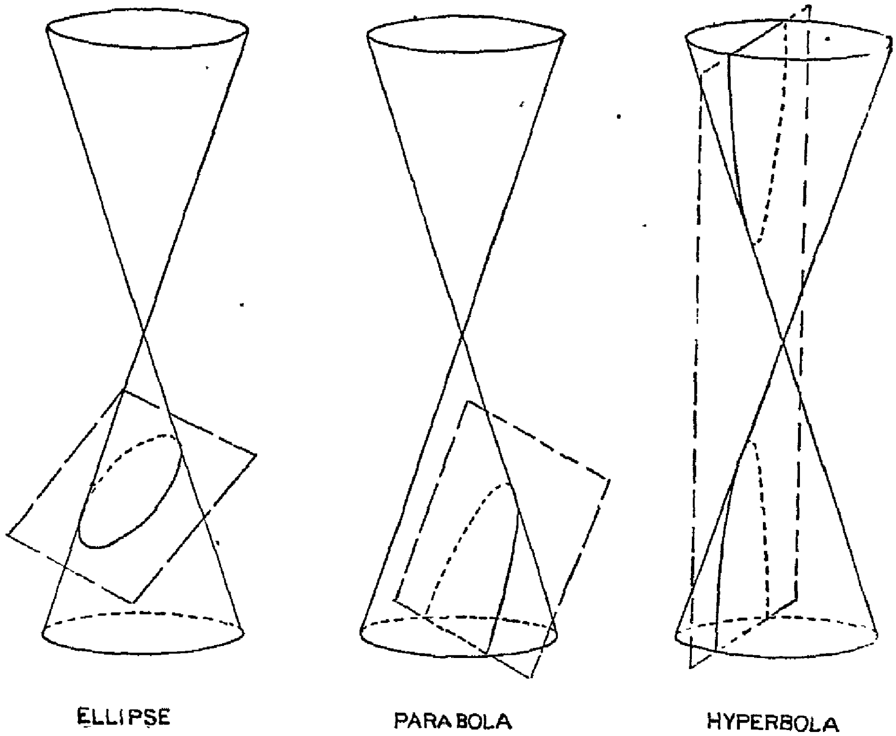


FIG. 10.1.

If the plane cuts across one nappe at an angle the curve of intersection is an ellipse. If the plane is parallel to an element of the cone the curve of intersection is a parabola. If the plane cuts both nappes the curve of intersection is a hyperbola.

The study of these curves is a part of plane analytic geometry. The graphs of these conics in plane analytic geometry, in the so-called "standard positions," are shown in Fig. 10.2.

In the application of conics to lofting, it happens that the curves in standard position are not very useful, and the basic theory as developed in plane analytic geometry is not the most

convenient approach. Instead, the viewpoint of synthetic and analytic projective geometry is adopted, and the graphical constructions and equations are based on Pascal's theorem and Brianchon's theorem.

Conics have many properties that make them admirably suited to the requirements of design and lofting. They are "fair" curves, and they can be selected so as to meet sets of conditions that arise frequently in design and lofting. They can be constructed graphically, and their equations can be cal-

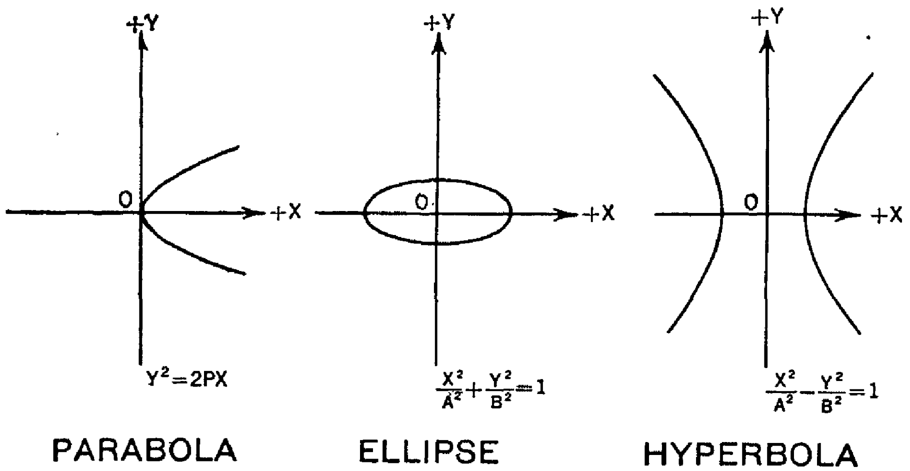


FIG. 10.2.

culated analytically. They can be reproduced easily and accurately. They can be used to approximate predetermined shapes. Their good streamline characteristics have been proved by performance.

This chapter will deal with graphical constructions and properties of conics from the projective geometry point of view, and applications to design and lofting.

10.1. Pascal's theorem. A generalized concept of a hexagon is that a hexagon is a figure formed by six points of a plane, no three of which lie in a straight line, and the lines joining these six points in any order (see Fig. 10.3).

Consider the hexagon 1-2-3-4-5-6, inscribed in a conic (see Fig. 10.4). The three pairs of opposite sides are 1-2, 4-5; 2-3, 5-6; 3-4, 6-1. Pascal's theorem states that the three pairs of opposite sides of a hexagon inscribed in a conic intersect in three points that lie on a straight line. In Fig. 10.4, 1-2, 4-5 intersect at P; 2-3, 5-6 intersect at Q; 3-4, 6-1 intersect at R. Pascal's theorem

states that P, Q, R lie on a straight line. This line is called *the Pascal line.* This theorem is the foundation for many useful geometrical constructions for conics. It is usually stated as follows:

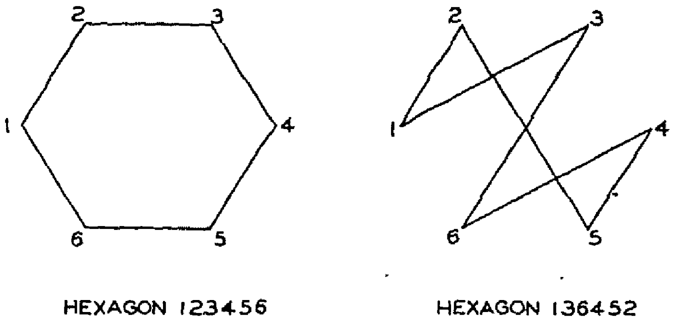
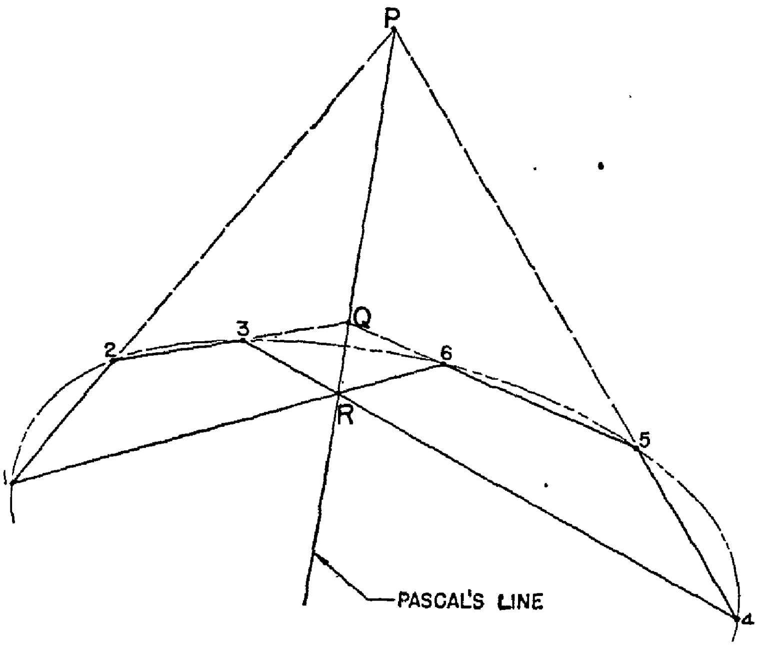


FIG. 10.3.



$\overline{1-2}$ AND $\overline{4-5}$ = P

$\overline{2-3}$ AND $\overline{5-6}$ = Q

$\overline{3-4}$ AND $\overline{6-1}$ = R

FIG. 10.4.

A necessary and sufficient condition that six points be points of a point conic is that the pairs of opposite sides of any simple hexagon having these points as vertexes meet in collinear points.

Since a circle is a conic, it is easy and instructive to test this theorem with a circle. Draw a circle with compasses. Select any six points on the circle and number the points 1, 2, 3, 4, 5, 6 in any order. List the three pairs of opposite sides. Draw these pairs of opposite sides and find their three points of intersection. Notice that these three points of intersection are collinear. Pascal proved his theorem for a circle and then deduced its truth for other conic sections from the fact that ellipses, parabolas, and hyperbolas can be obtained from circles by projections and sections.

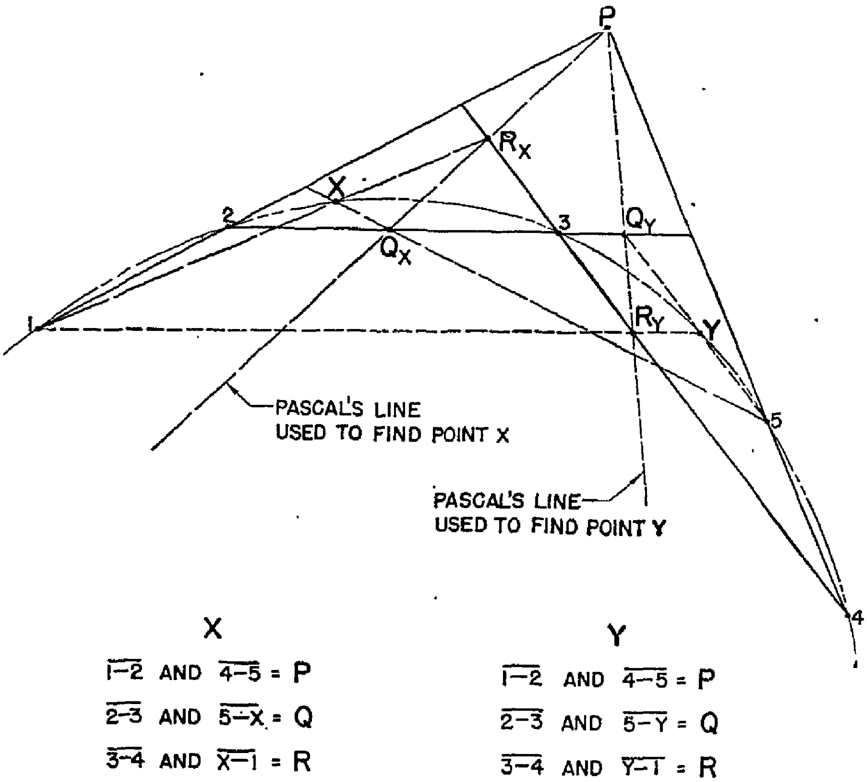


Fig. 10.5.

10.2. Conic determined by five points. Consider the conic determined by the five points 1, 2, 3, 4, 5 (see Fig. 10.5). Consider the problem of determining additional points on the given conic.

Let x be a sixth point on the conic. Then the Pascal hexagon is 1-2-3-4-5-x. The pairs of opposite sides are 1-2, 4-5; 2-3, 5-x; 3-4, x-1. Here 1-2 means the line through points 1 and 2, and similarly for the symbols 4-5, etc. Draw the three pairs of opposite sides, and find their points of intersection. The points

of intersection are P, Q, R. These three points determine the Pascal line.

The actual steps in the construction necessary to locate x, a sixth point on the conic, are:

1. Draw 1-2 and 4-5. These two lines intersect. Label the point of intersection P.
2. Through P draw any line. This is the Pascal line.
3. Draw 2-3. This line intersects the Pascal line. Label the point of intersection Q.
4. Draw 3-4. This line intersects the Pascal line. Label the point of intersection R.
5. Draw 5-Q and 1-R. These two lines intersect. Label the point of intersection x. This is the required point.

Notice that any number of points x can be found by changing the position of the Pascal line through P. See steps 1 and 2 above. In Fig. 10.5 another position of the Pascal line is shown, leading to the point y, another point on the conic.

10.3. Conic determined by one point slope and three additional points. Consider the conic determined by the five points 1, 2, 3, 4, 5. Suppose that points 1, 2 coincide. The line 1-2 will be a tangent to the conic at the point 2. In this special case we say that the conic is determined by the point slope 1-2 and the three additional points 3, 4, 5. The point and the tangent (slope) being two conditions, the total number of *conditions* is five, as in the case of the conic determined by five points (see Fig. 10.6).

To find a sixth point on the conic, proceed as follows:

1. Draw 4-5. The given tangent 1-2 and the line 4-5 intersect. Label the point of intersection P.
2. Through P draw any line. This is the Pascal line.
3. Draw 2-3. This line intersects the Pascal line. Label the point of intersection Q.
4. Draw 3-4. This line intersects the Pascal line. Label the point of intersection R.
5. Draw 5-Q and 1-R. These two lines intersect. Label the point of intersection x. This is the required point.

By varying the position of the Pascal line through P we can obtain additional points on the curve. The location of another point, y, is shown in Fig. 10.6.

The Pascal hexagon is 1-2-3-4-5-x. The pairs of opposite sides are 1-2, 4-5; 2-3, 5-x; 3-4, x-1. In this case 1-2 is the given

tangent to the conic. The points of intersection of the three pairs of opposite sides are P, Q, R. These three points determine the Pascal line.

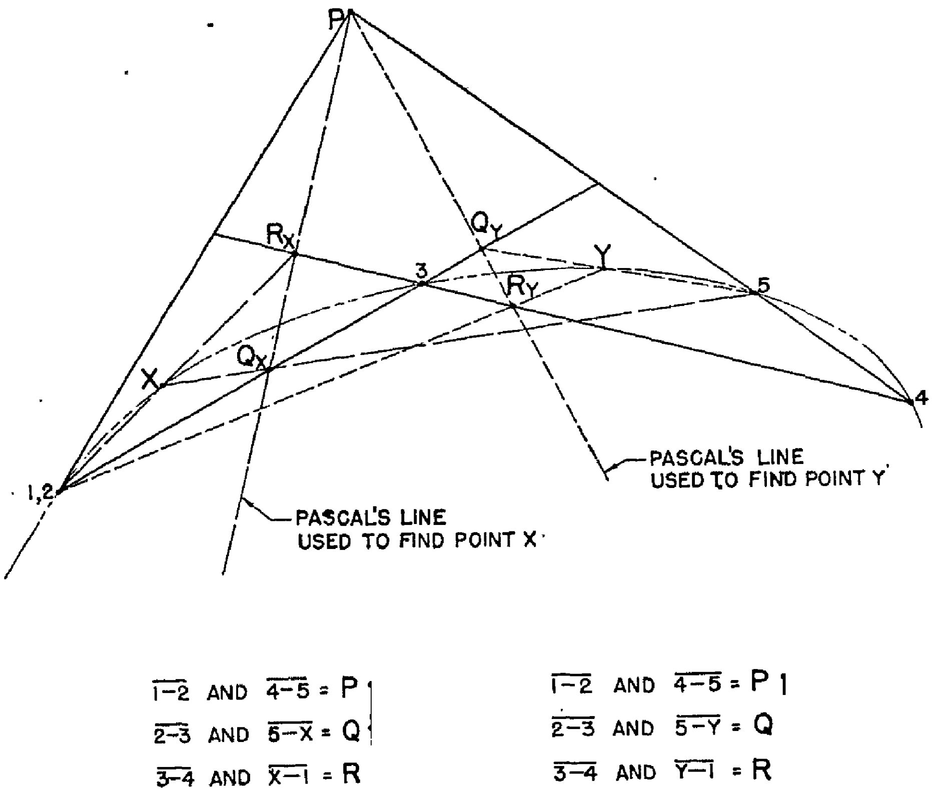


FIG. 10.6.

10.4. Conic determined by two point slopes and one additional point. Consider the conic determined by the five points 1, 2, 3, 4, 5. Suppose that points 1, 2 coincide and points 4, 5 coincide. The lines 1-2 and 4-5 will be two tangents to the conic, at points 2, 5 respectively. In this special case we say that the conic is determined by the point slope 1-2, the point slope 4-5, and the additional point 3. The point and the tangent (slope) being two conditions, the total number of *conditions* is five, as in the case of the conic determined by five points (see Fig. 10.7).

To find a sixth point on the conic, proceed as follows:

1. The given tangents 1-2 and 4-5 intersect. Label the point of intersection P.

2. Through P draw any line. This is the Pascal line.

3. Draw 2-3. This line intersects the Pascal line. Label the point of intersection Q.

4. Draw 3-4. This line intersects the Pascal line. Label the point of intersection R.

5. Draw 5-Q and 1-R. These two lines intersect. Label the point of intersection x. This is the required point.

By varying the position of the Pascal line through P we can obtain additional points on the curve.

The Pascal hexagon is 1-2-3-4-5-x. The pairs of opposite sides are 1-2, 4-5; 2-3, 5-x; 3-4, x-1. In this case 1-2 and 4-5 are

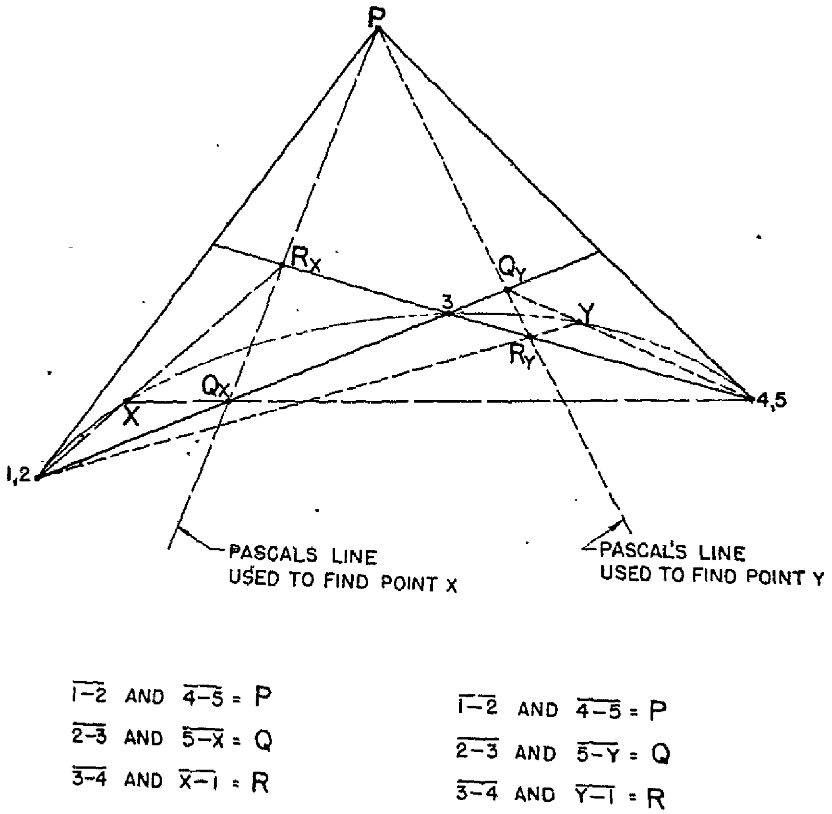


FIG. 10.7.

given tangents to the conic. The points of intersection of the three pairs of opposite sides are P, Q, R. These three points determine the Pascal line.

This is an extremely useful form of the Pascal construction. Very often the design requirements are that the tangents at two points be fixed in slope and that the curve pass through an additional (intermediate) point (see Fig. 10.8).

In designing the nose of this fuselage, the tangent at 1, 2 is required to be vertical, and the tangent at 4, 5 is fixed in slope

Also, the designed curve is to pass through the point 3. These conditions are exactly the "two point slopes and one additional point" described in this article. By means of the Pascal construction the design requirements can be met by a conic, which can be constructed geometrically by obtaining additional points as described in this article (compare Figs. 10.8 and 10.7).

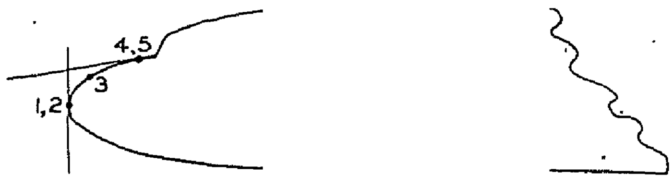


FIG. 10.8.

10.5. Tangent to a conic determined by five points. Consider the conic determined by the five points, 1, 2, 3, 4, 6. Consider the problem of determining the slope at one of the given points (see Fig. 10.9).

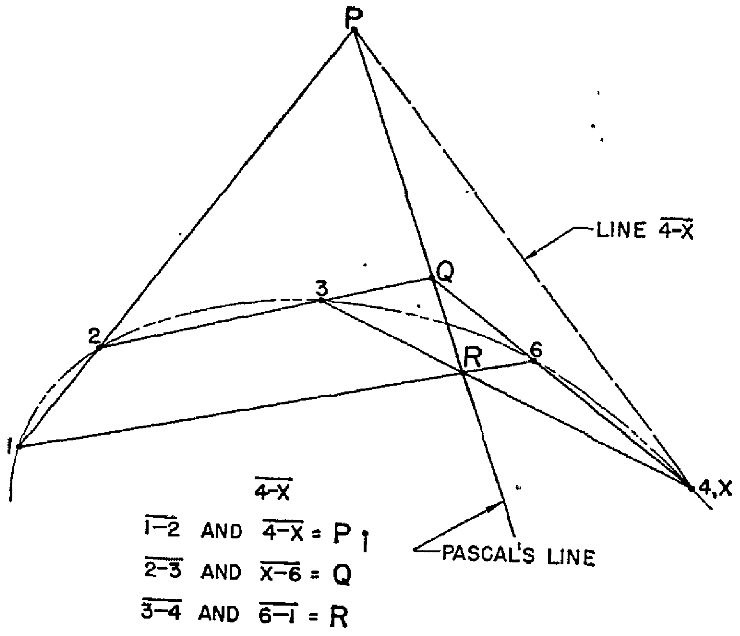


FIG. 10.9.

To find the tangent at 4, proceed as follows:

1. Draw 3-4 and 6-1. These two lines intersect. Label the point of intersection R.

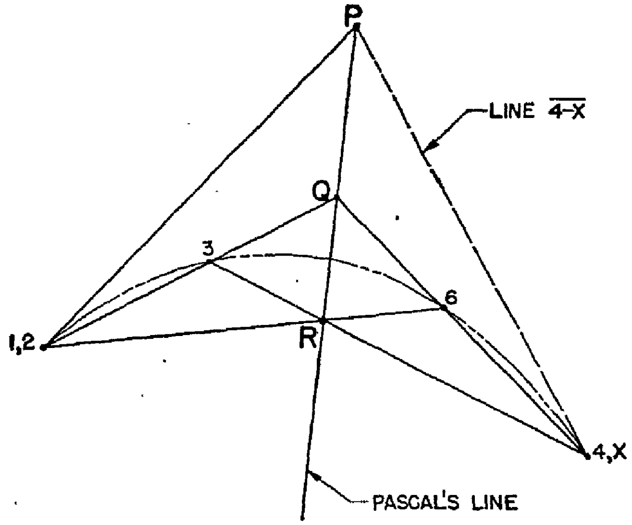
2. Draw x-6 and 2-3. These two lines intersect. Label the point of intersection Q.

3. Draw the line RQ. This is the Pascal line.

4. Draw 1-2. This line intersects the Pascal line. Label the point of intersection P.

5. Draw 4-P. This is the required tangent.

The Pascal hexagon is 1-2-3-4-x-6. The pairs of opposite sides are 1-2, 4-x; 2-3, x-6; 3-4, 6-1. These three pairs of opposite sides intersect at P, Q, R. The points 4, x coincide, and the line 4-x is the required tangent to the conic. The points P, Q, R determine the Pascal line.



$\overline{4-X}$

$\overline{1-2}$ AND $\overline{4-X}$ = P

$\overline{2-3}$ AND $\overline{X-6}$ = Q

$\overline{3-4}$ AND $\overline{6-1}$ = R

FIG. 10.10.

10.6. Tangent to a conic determined by one point slope and three additional points. Consider the conic determined by the point slope 1-2 and the three additional points 3, 4, 6. Consider the problem of determining the slope of the tangent at any one of the three additional points, say 4 (see Fig. 10.10).

To find the tangent at 4, proceed as follows:

1. Draw 3-4 and 6-1. These two lines intersect. Label the point of intersection R.

2. Draw 2-3 and 6-x. These two lines intersect. Label the point of intersection Q.

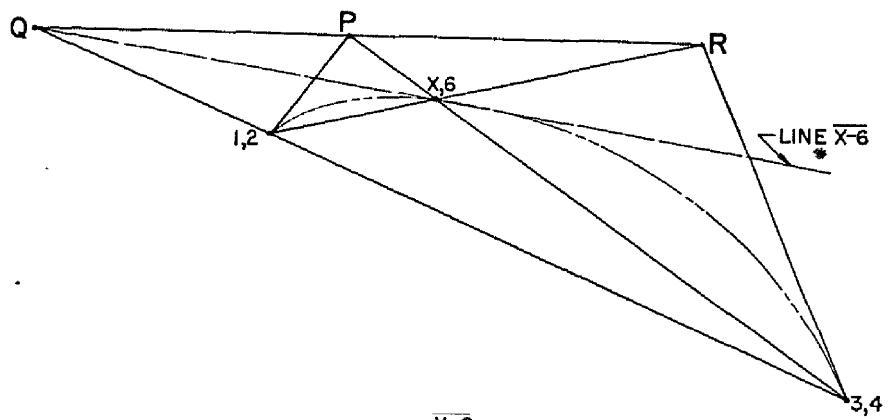
3. Draw RQ. This is the Pascal line.

4. The given tangent 1-2 intersects the Pascal line. Label the point of intersection P.

5. Draw 4-P. This is the required tangent.

The Pascal hexagon is 1-2-3-4-x-6. The three pairs of opposite sides are 1-2, 4-x; 2-3, x-6; 3-4, 6-1. These three pairs of opposite sides intersect in P, Q, R. The points P, Q, R determine the Pascal line.

10.7. Tangent to a conic determined by two point slopes and one additional point. Consider a conic determined by two point slopes 1-2 and 3-4, and one additional point, 6. Consider the



$\overline{X\text{-}6}$

$\overline{1\text{-}2}$ AND $\overline{4\text{-}X}$ = P

$\overline{2\text{-}3}$ AND $\overline{X\text{-}6}$ = Q

$\overline{3\text{-}4}$ AND $\overline{6\text{-}1}$ = R

Fig. 10.11.

problem of determining the tangent at the additional point, 6 (see Fig. 10.11).

To find the tangent at 6, proceed as follows:

1. Draw the line 6-1. The line 6-1 and the tangent 3-4 intersect. Label the point of intersection R.

2. Draw the line 4-x. The line 4-x and the tangent 1-2 intersect. Label the point of intersection P.

3. Draw PR. This is the Pascal line.

4. Draw the line 2-3. The line 2-3 intersects the Pascal line. Label the point of intersection Q.

5. Draw the line 6-Q. This is the required tangent.

The Pascal hexagon is 1-2-3-4-x-6. The three pairs of opposite sides are 1-2, 4-x; 2-3, x-6; 3-4, 6-1. These three pairs of opposite

sides intersect in the three points P, Q, R. The points P, Q, R determine the Pascal line.

This construction gives the tangent at the control point, 6.

10.8. The control point. When a conic is determined by two point slopes and an additional point, the additional point serves as a control point which determines the shape of the conic. By

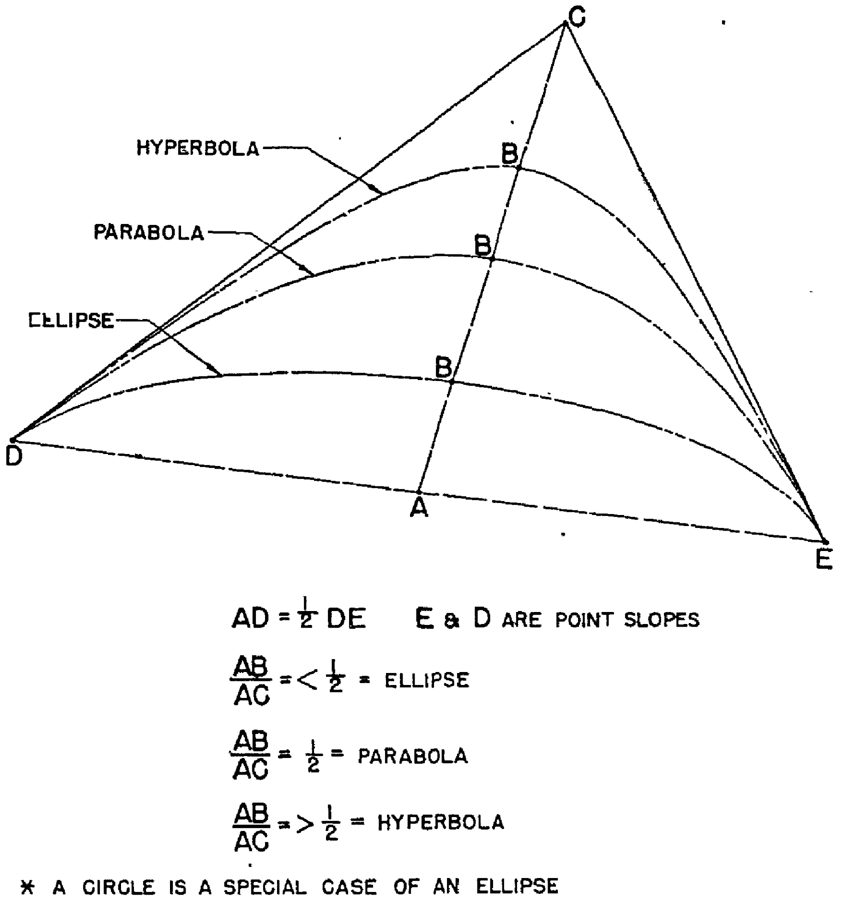


FIG. 10.12.

varying the position of the control point an infinite variety of shapes can be determined, each of which continues to have the same fixed point slopes (see Fig. 10.12).

In Fig. 10.12, the two given fixed points and slopes are D, DC; E, EC. The point B is the control point and is shown in several different positions. Draw the chord DE and draw CA from C to A, where A is the mid-point of the chord DE. Consider the ratio $\frac{AB}{AC}$. If $\frac{AB}{AC} = \frac{1}{2}$, then the conic is a parabola. If $\frac{AB}{AC} < \frac{1}{2}$,

the conic is an ellipse. If $\frac{AB}{AC} > \frac{1}{2}$, the conic is a hyperbola. Notice that a circle is a special case of an ellipse. If we always select a control point B on the line CA, the curve can be precisely specified by stating the exact value of AB/AC. It is not necessary, however, to restrict the control point to the line AC. The control point can be anywhere inside the triangle CDE. This arbitrariness in the position of the control point is one of the best features of this type of curve as applied in designing and lofting.

10.9. Brianchon's theorem. From the standpoint of projective geometry, there are point conics and line conics. Point

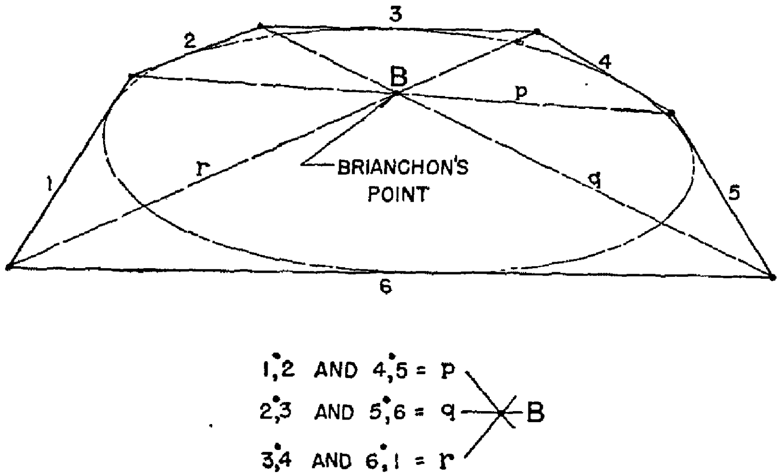


FIG. 10.13.

conics are determined by points and line conics are determined by tangents (lines). The line conic analogue of Pascal's theorem is Brianchon's theorem. This theorem may be stated as follows:

If the sides of a hexagon are tangents to a conic, the lines joining pairs of opposite vertexes pass through one point.

Brianchon's theorem gives rise to a set of graphical constructions similar to those resulting from Pascal's theorem. It is used to find additional lines tangent to a conic when the conic is determined by five lines (tangents) and to solve many special cases and variations of this basic problem.

In Fig. 10.13, the conic is inscribed in the hexagon 1-2-3-4-5-6. The three pairs of opposite vertexes are the points of intersection of 1, 2 and 4, 5; 2, 3 and 5, 6; 3, 4 and 6, 1. These three pairs of opposite vertexes determine the lines p, q, r. The lines p, q, r determine the Brianchon point B.

10.10. Conic determined by five lines (tangents). Consider the conic determined by five lines, no three of which are concurrent (see Fig. 10.14). Consider the problem of finding additional tangents to the given conic.

In Fig. 10.14, the conic is determined by the five lines (tangents) 1, 2, 3, 4, 5. The notation 1, 2 indicates the point of intersection of lines 1 and 2, and similarly for 2, 3, etc. The three

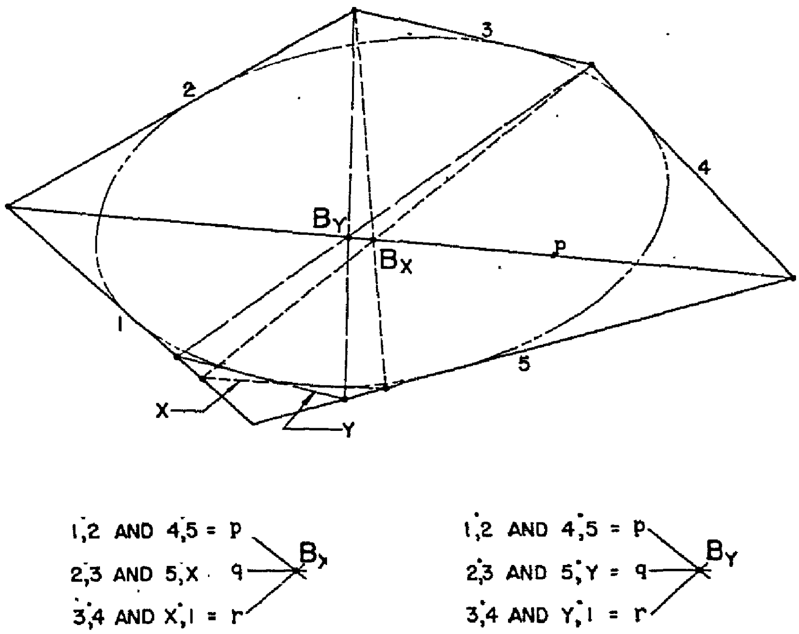


FIG. 10.14.

pairs of opposite vertexes are

1, 2 and 4, 5
2, 3 and 5, x
3, 4 and x, 1.

The line joining the two points 1, 2 and 4, 5 is p. The line joining the two points 2, 3 and 5, x is q. The line joining the two points 3, 4 and x, 1 is r. The lines p, q, r intersect in a point, B. The point B is the Brianchon point of the hexagon 1-2-3-4-5-x.

To construct the additional tangent x to the conic determined by the five tangents 1, 2, 3, 4, 5, proceed as follows:

1. Join the point 1, 2 to the point 4, 5. Label this line p.

2. On the line p select any point B. This is the Brianchon point.

3. Join the point 2, 3 to the point B. Label this line q.

4. Join the point 3, 4 to B. Label this line r.

5. The lines 5 and q and the lines 1 and r determine two points of intersection which in turn determine an additional tangent to the given conic.

By varying the position of the arbitrarily selected point B, additional tangents to the conic can be determined. In Fig. 10.14, the construction for locating another tangent, y, is shown.

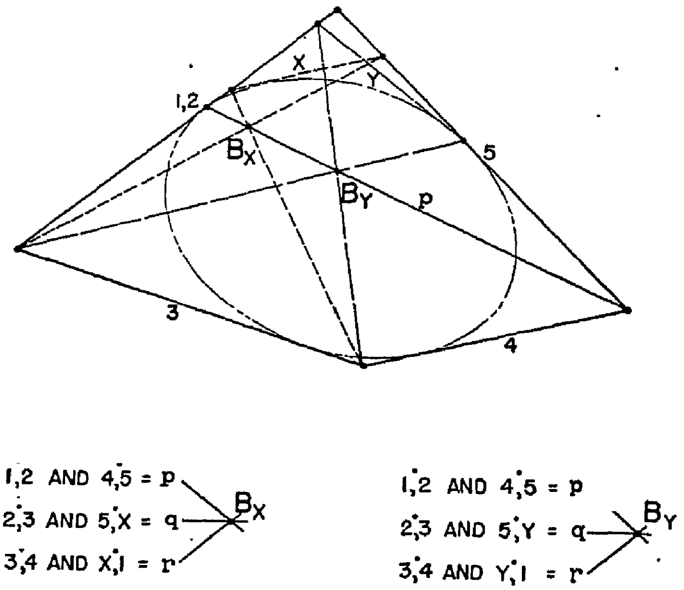


FIG. 10.15.

10.11. Special cases of Brianchon's theorem. Brianchon's theorem can be specialized to solve the following problems:

1. To find additional lines tangent to the conic determined by one slope point and three additional lines (tangents).

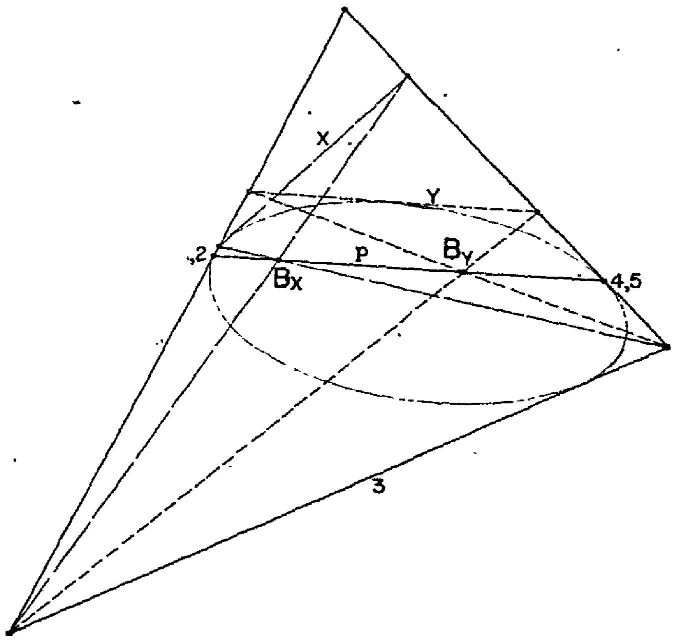
2. To find additional lines tangent to the conic determined by two slope points and one additional line (tangent).

3. To find the point of contact (tangent point) on one of the five tangents to a conic determined by five given tangents.

4. To find the point of contact (tangent point) on one of the three tangents to a conic determined by one slope point and three additional lines (tangents).

5. To find the point of contact (tangent point) on the tangent to a conic determined by two slope points and one additional line (tangent).

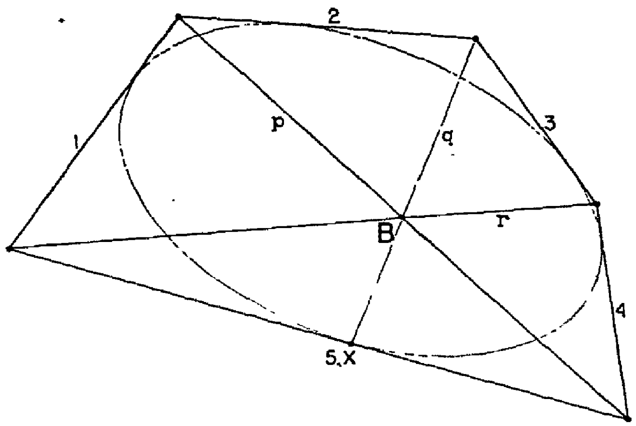
These problems and their solutions are given in Figs. 10.15 to 10.19, inclusive. The notations are those adopted in Art. 10.10.



1,2 AND 4,5 = p
2,3 AND 5,X = q
3,4 AND X,1 = r
B_X

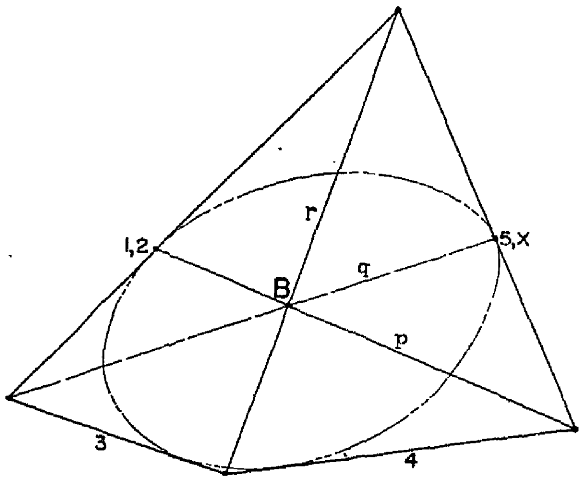
1,2 AND 4,5 = p
2,3 AND 5,Y = q
3,4 AND Y,1 = r
B_Y

FIG. 10.16.



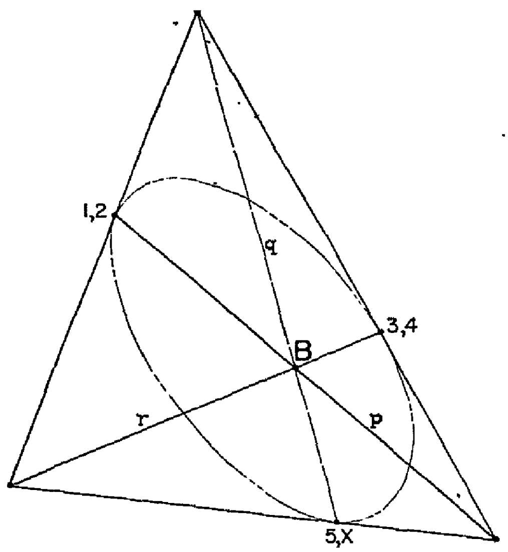
5,X
1,2 AND 4,5 = p
2,3 AND 5,X = q
3,4 AND X,1 = r
B

FIG. 10.17.



5,X
1,2 AND 4,5 = p
2,3 AND 5,X = q — B
3,4 AND X,1 = r

Fig. 10.18.

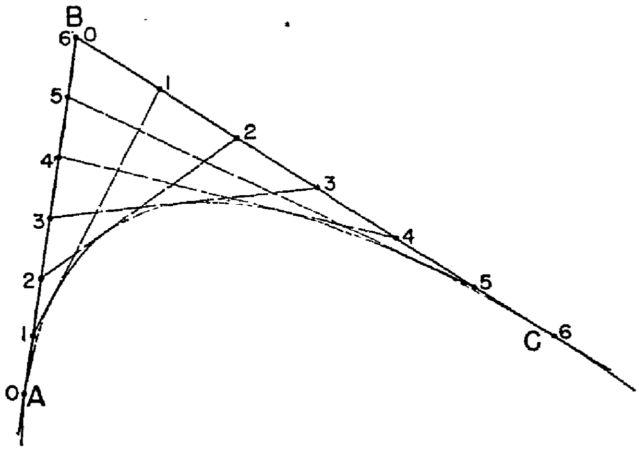


5,X
1,2 AND 4,5 = p
2,3 AND 5,X = q — B
3,4 AND X,1 = r

Fig. 10.19.

The constructions are specialized from the general construction in Art. 10.10 in much the same way that the special cases of Pascal's theorem were derived in Arts. 10.3 to 10.7, inclusive.

10.12. Tangent construction for a parabola. The construction illustrated in Fig. 10.20 is one that finds many applications in lofting and design. In Fig. 10.20, the lines AB and CB are given fixed tangents. A curve is desired that will be tangent to AB at A and tangent to CB at C. No other limitation is to be placed on the curve. A parabola is a curve that will fit these prescribed



LINES AB AND CB ARE DIVIDED INTO SAME NUMBER OF EQUAL PARTS

FIG. 10.20.

conditions. To construct the parabola by tangents, proceed as follows:

1. Divide AB into any convenient number of equal parts, say 6.
2. Divide CB into the *same* number of equal parts. Notice that the parts of CB are equal to each other, but they are not equal to the parts of AB unless $AB = CB$.
3. Join the corresponding points of division, as shown on Fig. 10.20. These lines are each tangent to the required curve.

Parabolas of this type are similar to vertical curves used in railroad engineering.

To establish the exact points of tangency, make use of the following relations:

1. The distance from the point 1 on AB to the point of tangency on 1-1 is $\frac{1}{6}$ of the total length of 1-1.

2. The distance from the point 2 on AB to the point of tangency on 2-2 is $\frac{2}{6}$ of the total length of 2-2.
3. The distance from the point 3 on AB to the point of tangency on 3-3 is $\frac{3}{6}$ of the total length of 3-3.
4. The distance from the point 4 on AB to the point of tangency on 4-4 is $\frac{4}{6}$ of the total length of 4-4.
5. The distance from the point 5 on AB to the point of tangency on 5-5 is $\frac{5}{6}$ of the total length of 5-5.

Similar relations hold when the segments AB, CB are divided in any number of equal parts. In general, if the segments AB, CB are each divided into n equal parts, and if we consider the kth tangent, then the distance from point k on AB to the point of tangency on k-k is $\frac{k}{n}$ times the total length of k-k.

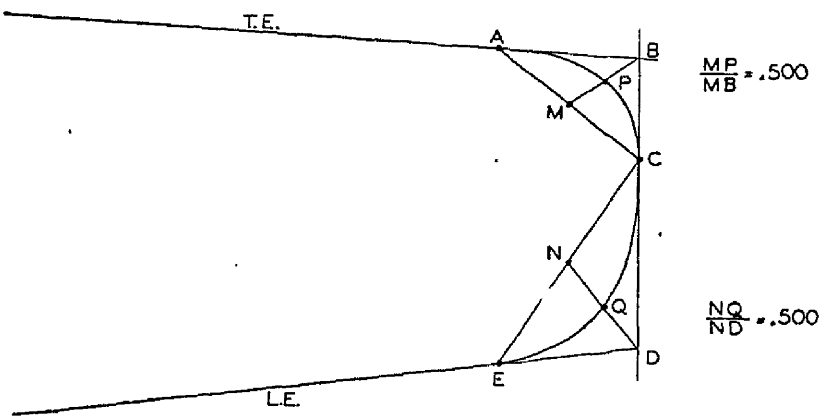


FIG. 10.21.

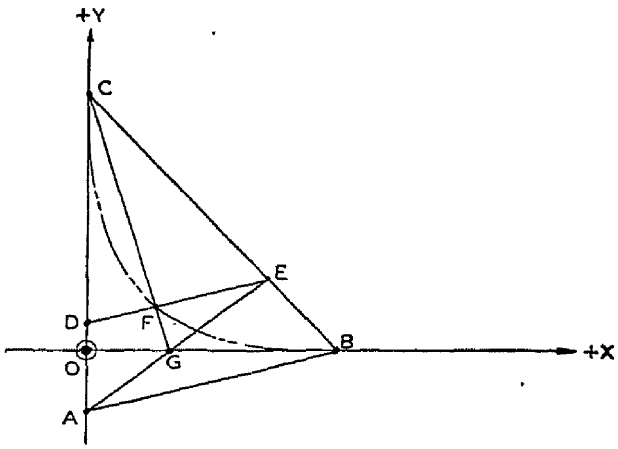
Since the curve in Fig. 10.20 is a parabola, the control point may be found as follows:

1. Draw the chord AC.
2. Find the mid-point of AC and label it M.
3. Draw the line segment BM.
4. Find the mid-point of BM and label it P.

This is the control point, and $PM/BM = \frac{1}{2}$. This parabola can be used whenever a curve is required which is to be tangent to AB at A, tangent to CB at C, and the curve can be of a shape that will allow it to pass through the point P as described above. Instead of using the tangent lines construction, the point P can be used as a control point, and the construction of Art. 10.4 can be used.

An example of such a set of requirements is shown in the wing tip in Fig. 10.21, where two parabolas are used to design the wing-tip curve. In Fig. 10.21, the given fixed tangents are AB, BD, DE, and the given fixed points of tangency are A, C, E. The point M is the mid-point of AC, and P is the mid-point of BM. The point N is the mid-point of CE, and Q is the mid-point of DN.

10.13. A hyperbola construction. Consider the problem of constructing a curve that shall be tangent to OB at B, and tangent to OC at C (see Fig. 10.22). These conditions can be met by a parabola, constructed by the tangents method or by the control point method. In Fig. 10.22 there is still another



HYPERBOLA

FIG. 10.22.

solution to the problem. This construction yields a hyperbola. The procedure is as follows:

1. Draw AB at any angle to OB.
2. Draw DE parallel to AB.
3. Draw AE. This line intersects OB. Label the point of intersection G.
4. Draw CG.
5. The lines CG, DE intersect. Label the point of intersection F. This is a point on the required curve.

Additional points can be located by varying the position of DE parallel to AB. Parallel rulers are useful in drawing DE parallel to AB.

The line AB is a control line. Once the position of AB has been established, the shape of the curve is uniquely determined.

Different shapes can be obtained by varying the position of the control line AB. All these shapes will be tangent to the x axis at B and tangent to the y axis at C.

If $OC = OA$, the curve is a parabola.

10.14. Construction of a parabola by tangents (alternative method). An alternative method for constructing a parabola by tangents is shown in Fig. 10.23. The parabola is to be tangent

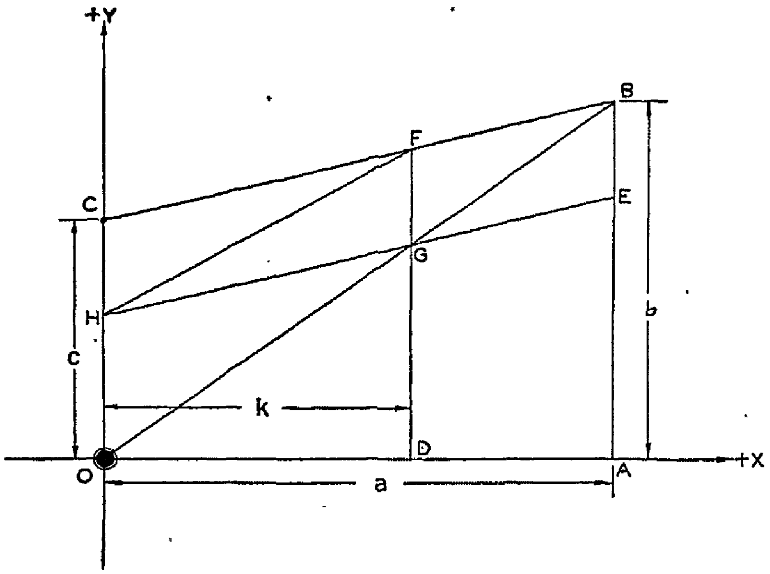


FIG. 10.23.

to the y axis at the origin, and tangent to the line BC at B (see Fig. 10.23).

The construction is as follows:

1. Select any point on OA, say D.
2. Draw DF parallel to OC.
3. Draw the chord OB.
4. Lines OB, DF intersect at G.
5. Through G draw HE parallel to BC.
6. Lines HE, OC intersect at H.
7. Lines DF, BC intersect at F.
8. Line HF is tangent to the parabola. By varying the position of the point D on OA, other tangents can be found. The parabola may be constructed by setting a spline tangent to this envelope of tangents.

To construct a series of tangents, parallel rulers can be used to expedite the construction. Draw OB first. Use the parallel

rulers to mark a series of points such as G and F. Then use the parallel rulers to mark a series of points H (see Fig. 10.24). The lines G_iF_i are parallel to the tangent OC, and the lines H_iG_i are

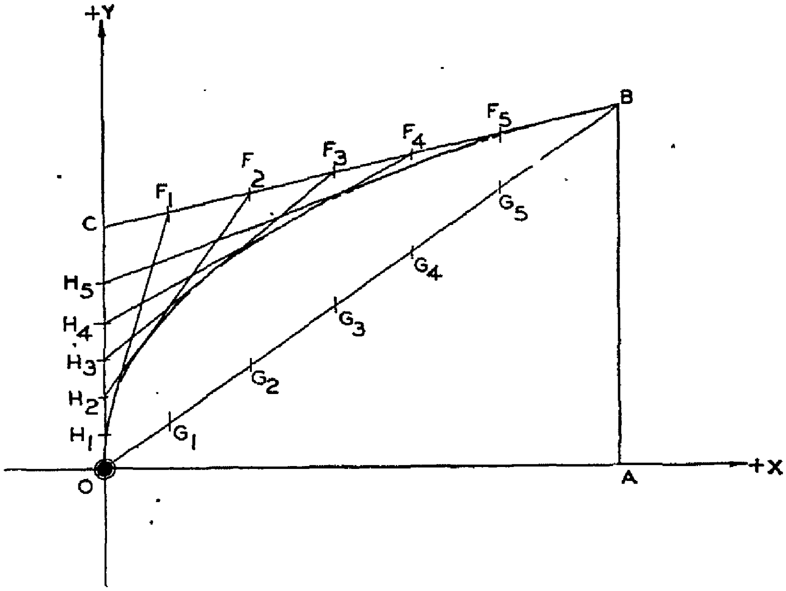


FIG. 10.24.

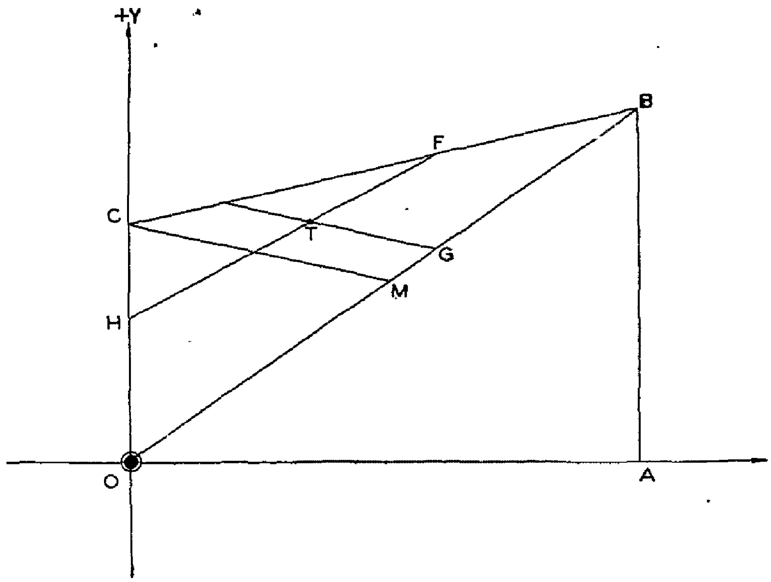


FIG. 10.25.

parallel to the tangent BC. The lines H_iF_i are the required tangents.

The exact points of tangency may be found as follows. Locate M, the mid-point of OB. Draw CM. Lines GT drawn through

G parallel to CM will intersect the tangents FH at the points of tangency (see Fig. 10.25).

If the tangent FH is required to yield a certain predetermined point of tangency, say at $x = p$, choose D at $x = \sqrt{ap}$, so that $k = \sqrt{ap}$. See Fig. 10.23.

10.15. Lofting by conics. Many different factors enter into the determination of the outside contours of an airplane. After much preliminary investigation the final shape must be completely and precisely described before mass production can begin. The manner of this description may assume many different forms: lines, numbers, drawings, tables of offsets, sample parts, templates, master patterns, dimensions of tools, etc. The more mathematical the description is, the more the fabrication process may be broken up and distributed among different tools and different processes. The shipbuilding industry years ago began to demand and seek precise shapes by water line, buttock line, frame station, diagonal line fairing processes known as ship lofting. Later the automobile industry and the airplane industry adopted these methods, with certain variations. The advantages of predetermining precise forms include interchangeability, standardization, and manufacturing breakdowns. The result is increased production.

The use of tables of offsets and full-scale frame lines is common in shipbuilding; the use of solid patterns is common in automobile building; the use of loft layouts and plaster patterns is common in aircraft tooling in developing cowls, ducts, fillets, and other complex shapes. There are certain disadvantages inherent in loft layouts and plaster patterns. From an analytical standpoint they are indeterminate. If the loft layouts or plaster patterns were destroyed, new ones would not be exact duplicates of the original. In order to convey a description of a part to another location it is necessary actually to transport the part or pattern. Many engineering shapes are so complicated that these methods are the only ones feasible. On the other hand, a surprisingly large variety of shapes can be described completely mathematically—by formulas and by geometrical constructions.

Conic sections are well suited to the requirements of lofting and design. Both the graphical and analytical methods are useful. An illustration of how conics may be used in lofting is shown in Fig. 10.26.

In Fig. 10.26, the profile view shows the curve of intersection made by the plane of symmetry ($x = 0$) with the nose of the fuselage. This curve of intersection is determined by two conics, an upper curve and a lower curve. Each is determined by a horizontal tangent, a vertical tangent, the points of tangency, and a control point. The equations of these curves are calculated.

The body plan view shows the curves at fuselage stations, made by the intersection of planes of the type y = constant with the nose of the fuselage.

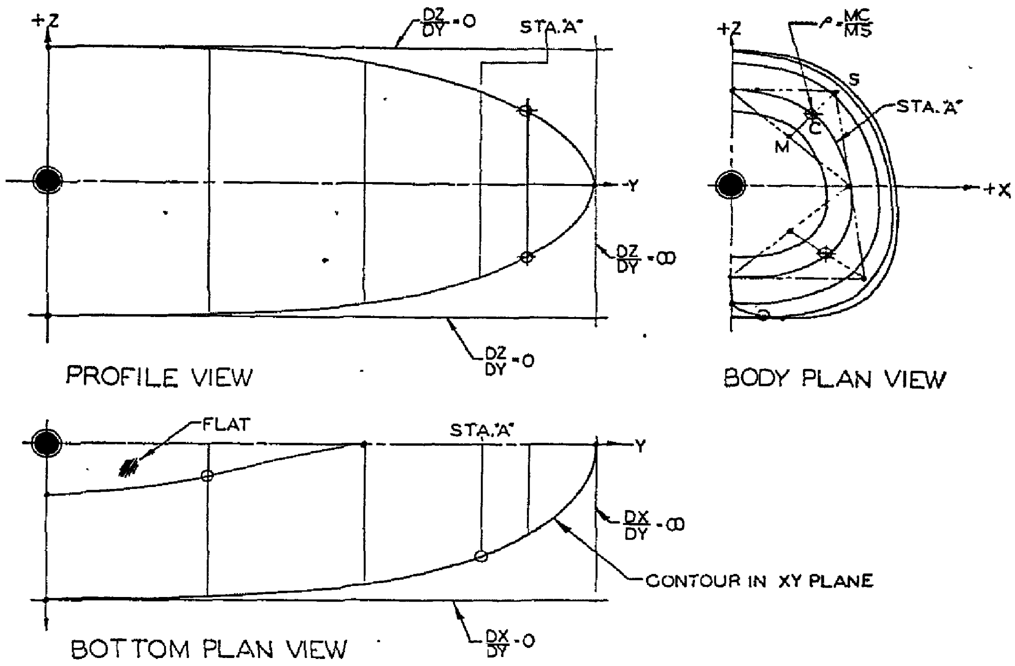


FIG. 10.26.

The bottom plan view shows the maximum half breadth curve, which is a conic determined by two point slopes and a point. The equation of this curve is calculated.

To determine the curves of the cross section at fuselage station A, the equations of the curves in the profile view and the bottom plan view are used to determine the upper, lower, and xy-plane width points. The slopes of the tangents are fixed as shown in Fig. 10.26. The control points in the upper and lower conics of the body plan view are determined by assigning a ρ value for the body plan curves.

The body plan conics are constructed graphically. Notice that each is determined by two point slopes and a control point, and that these are determined mathematically.

The shape of the surface is therefore determined in a precise manner, so that the three basic views are mathematically related. The profile view and bottom plan view are not drawn full size. The mathematical equations suffice for their determination. The body plan view is drawn full size as a loft layout, but the usual water line, buttock line, frame line, and diagonal line fairing procedure is eliminated.

CHAPTER 11

CONICS. ANALYTICAL THEORY

Conic sections are useful when treated analytically as well as when treated graphically. For long curves the graphical methods for laying out conics meet with physical difficulties and limitations, which can be overcome by writing equations for the curves and then plotting the curve by offsets (ordinates). The graphical methods are more useful in preliminary design, scale layouts, and the lofting of fillets, wing-tip cross sections and body plan sections of fuselage and nacelle shapes on small airplanes. Curves that are used as longitudinal control lines and do not appear on any template, such as fuselage top and bottom profile, maximum half breadth and side control lines, may be determined advantageously by calculation rather than layout.

This chapter will include the derivation of formulas for conic sections, and applications of these formulas to practical situations as they arise in design and lofting.

11.1. General equation of a conic. The general equation of the second degree is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

This equation represents a conic section. The value of $B^2 - 4AC$ determines the nature of the conic, as follows:

$$\begin{array}{ll} B^2 - 4AC < 0. & \text{ellipse} \\ B^2 - 4AC = 0. & \text{parabola} \\ B^2 - 4AC > 0. & \text{hyperbola} \end{array}$$

The general equation appears to have six constants, but in reality has only five arbitrary constants, since division by one of them, say A, gives

$$x^2 + \frac{B}{A}xy + \frac{C}{A}y^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0,$$

which contains five undetermined coefficients. Therefore five independent conditions are sufficient to determine a conic. The

most convenient method for determining the equation of a conic from five given conditions is based on the theory of the degenerate base conics of a pencil of conics.

11.2. Conic determined by five points, no three of which are collinear. Consider the problem of finding the equation of a conic through five given points, no three of which lie in a straight line. Let the points be 1, 2, 3, 4, 5 (see Fig. 11.1). We shall

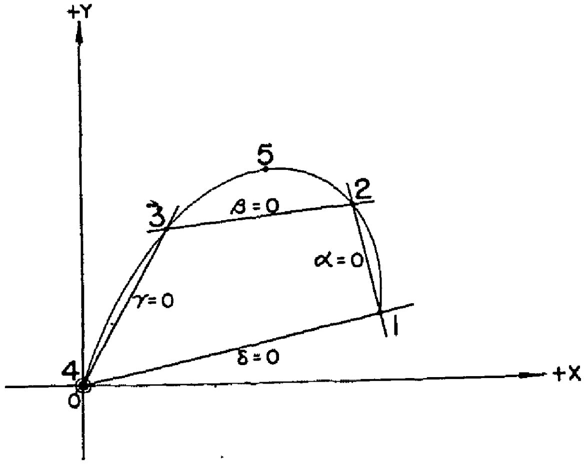


FIG. 11.1.

restrict our attention to the case in which the conic goes through the origin, with an equation of the type

$$Ay^2 + Bxy + Dx^2 + Ey + Fx = 0.$$

Write the equations of the lines 4-3, 2-1, 3-2, 4-1. Let these equations be denoted by

$$\begin{aligned} 4\text{-}3 &: \gamma = 0. \\ 2\text{-}1 &: \alpha = 0. \\ 3\text{-}2 &: \beta = 0. \\ 4\text{-}1 &: \delta = 0. \end{aligned}$$

Then the equation

$$K\beta\delta + \alpha\gamma = 0$$

represents the family of conics through the points 4, 3, 2, 1. In this equation K is a parameter, and its value can be determined from the fifth condition, which is that the conic pass through the point 5. Notice that each of the equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ is a first-degree equation in x and y, and so the equation $K\beta\delta + \alpha\gamma = 0$ is an equation of the second degree in x and y.

The point 4 is the intersection of the two straight lines $\gamma = 0$ and $\delta = 0$. The coordinates of 4 therefore satisfy the equations $\gamma = 0$ and $\delta = 0$, *i.e.*, they reduce the left-hand side of $\gamma = 0$ and $\delta = 0$ to zero. Since we have, then, that $\gamma = 0$ and $\delta = 0$, the equation $K\beta\delta + \alpha\gamma = 0$ becomes $K\beta(0) + \alpha(0) = 0$, or $0 = 0$. Therefore the coordinates of 4 satisfy the equation $K\beta\delta + \alpha\gamma = 0$, and the point 4 lies on the locus represented by $K\beta\delta + \alpha\gamma = 0$. Likewise, it can be shown that 3, 2, 1 lie on the locus represented by $K\beta\delta + \alpha\gamma = 0$.

Example. Find the equation of the conic through the five points 4(0, 0), 3(1, 2), 2(2, 3), 1(4, 4), 5(7,5).

The equations of 4-3, 2-1, 3-2, 4-1 are

$$\begin{aligned} \gamma\!:\!4\text{-}3\!:\!y - 2x &= 0. \\ \alpha\!:\!2\text{-}1\!:\!x - 2y + 4 &= 0. \\ \beta\!:\!3\text{-}2\!:\!x - y + 1 &= 0. \\ \delta\!:\!4\text{-}1\!:\!y - x &= 0. \end{aligned}$$

The equation $K\beta\delta + \alpha\gamma = 0$ becomes

$$K(x - y + 1)(y - x) + (x - 2y + 4)(y - 2x) = 0.$$

To evaluate K, let $x = 7$ and $y = 5$. Here (7, 5) are the coordinates of point 5. We obtain

$$\begin{aligned} K(7 - 5 + 1)(5 - 7) + (7 - 10 + 4)(5 - 14) &= 0. \\ -6K - 9 &= 0. \\ K &= -1.5. \end{aligned}$$

Substitute this value for K in the equation above.

$$\begin{aligned} -1.5(x - y + 1)(y - x) + (x - 2y + 4)(y - 2x) &= 0. \\ -0.5x^2 + 2xy - 0.5y^2 + 2.5y - 6.5x &= 0. \\ x^2 - 4xy + y^2 - 5y + 13x &= 0. \end{aligned}$$

This is the equation of the conic through the five given points. Substituting the coordinates of each of the given points in the equation, we find that all five points satisfy this equation.

The answer obtained for this example is in the general form. This is not the most convenient form for calculating offsets (ordinates). In order to obtain this form, arrange the terms so as to form a quadratic equation in y:

$$y^2 + y(-4x - 5) + (x^2 + 13x) = 0.$$

Solving this by the quadratic formula,

$$y \qquad 4x + 5 \pm \sqrt{(-4x - 5)^2 - 4(x^2 + 13x)}$$

$$y = 2x + 2.5 \pm \sqrt{3x^2 - 3x + 6.25}.$$

When the equation of the conic is in this form, x represents the station distance (abscissa) and y represents the offset to the curve (ordinate). The proper choice of the $\pm$ sign can be made by checking the equation by substituting the coordinates of one of the given points. In this example, $x = 1$ yields $y = 2$ if the minus sign is used for the radical, and so the final result is

$$y = 2x + 2.5 - \sqrt{3x^2 - 3x + 6.25}.$$

In order to simplify the calculation as much as possible, it is convenient to write the equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ in the form

$$\begin{aligned} \alpha: y - m_1x - h_1 &= 0, \\ \beta: y - m_2x - h_2 &= 0, \\ \gamma: x - m_3y &= 0, \\ \delta: y - m_4x &= 0. \end{aligned}$$

Substitute these equations in the formula $K\beta\delta + \alpha\gamma = 0$. We obtain

$$K(y - m_2x - h_2)(y - m_4x) + (y - m_1x - h_1)(x - m_3y) = 0.$$

Simplify by multiplying and collecting terms.

$$\begin{aligned} &(K - m_3)y^2 + [1 + m_1m_3 - K(m_2 + m_4)]xy \\ &\quad + (Km_2m_4 - m_1)x^2 + (h_1m_3 - h_2K)y + (h_2m_4K - h_1)x = 0. \end{aligned}$$

Compare this result with the general equation

$$Ay^2 + Bxy + Dx^2 + Ey + Fx = 0.$$

By equating coefficients we obtain general formulas for A, B, D, E, and F. Now solve the equation for y in terms of x. Assume the final result to be

$$y = cx + d \pm \sqrt{ax^2 + bx + d^2}.$$

We find the following general values for the coefficients:

$$\begin{aligned} A &= K - m_3. \\ B &= 1 + m_1m_3 - K(m_2 + m_4). \\ D &= Km_2m_4 - m_1. \\ E &= h_1m_3 - h_2K. \\ F &= h_2m_4K - h_1. \\ c &= \frac{\cdot B}{2A} \end{aligned}$$

$$d = \frac{-E}{2A}$$

$$a = c^2 - \frac{D}{A}.$$

$$b = 2cd -$$

Instead of re-solving the problem each time it occurs, we can use these formulas to calculate the final result immediately.

To illustrate the procedure, calculate the final result for the preceding example from these formulas.

$$\alpha = y - \tfrac{1}{2}x - 2.$$
$$\beta = y - x - 1.$$
$$\gamma = x - \tfrac{1}{2}y.$$
$$\delta = y - x.$$

$$m_1 = \tfrac{1}{2}, \quad m_2 = 1, \quad m_3 = \tfrac{1}{2}, \quad m_4 = 1, \quad h_1 = 2, \quad h_2 = 1.$$

$$K = -\frac{\alpha\gamma}{\beta\delta} = 0.375 \text{ (substituting } x = 7,\ y = 5).$$

$$\begin{aligned} A &= -0.125. & c &= 2. \\ B &= 0.5 & d &= 2.5. \\ D &= -0.125. & a &= 3. \\ E &= 0.625. & b &= -3. \\ F &= -1.625. \end{aligned}$$

$$y = cx + d \pm \sqrt{ax^2 + bx + d^2}.$$
$$y = 2x + 2.5 - \sqrt{3x^2 - 3x + 6.25}.$$

This result checks with the result obtained in the example on page 225.

Sometimes it is more convenient to calculate the x coordinates from given y coordinates. In this case,

$$x = cy + d \pm \sqrt{ay^2 + by + d^2},$$

where

$$c = \frac{-B}{2D},$$

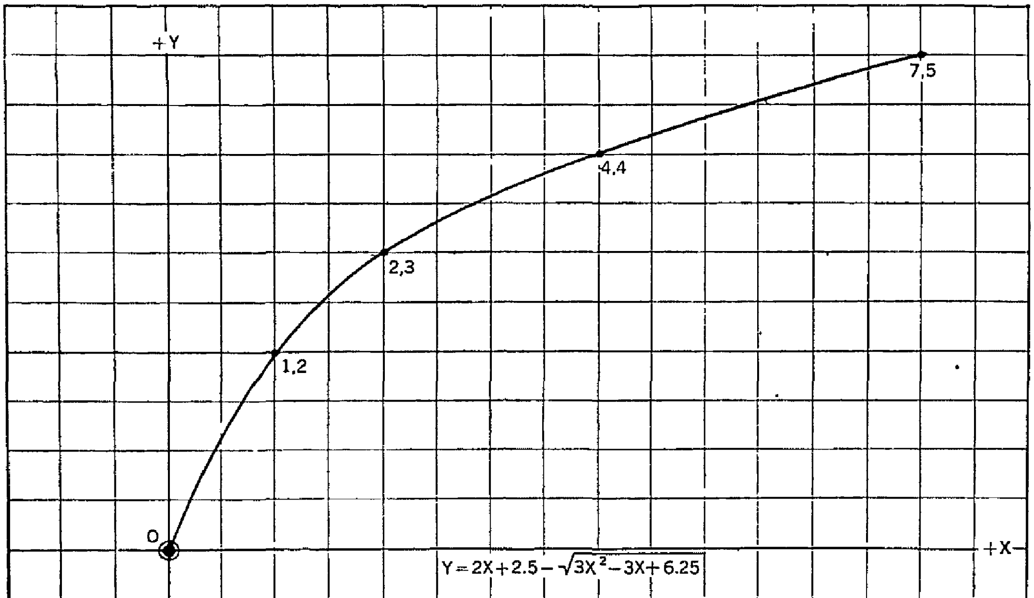
$$d = \frac{-F}{2D},$$

$$a = c^2 - \frac{A}{D},$$

$$b = 2cd - \frac{E}{D}.$$

These formulas can be obtained from the other case by interchanging the coefficients of x and y (interchanging A and D, and E and F). The values of the constants m_1, m_2, m_3, m_4, h_1, h_2, A, B, D, E, F are the same as in the previous case, when the equation was solved for y in terms of x.

When the equation of a curve is $y = f(x)$, given values of x can be substituted in the equation, and the corresponding values



X	0	0 5	1	1 5	2	2.5	3	3.5	4	4.5	5	5 5	6	6.5	7
Y	0	1.155	2	2.585	3	3.317	3.576	3.799	4	4.186	4.361	4 528	4.689	4 846	5

FIG. 11.2.

of y can be calculated. A table of offsets to the curve can be prepared, and this table can be used to lay out the curve (see Fig. 11.2).

The curve in Fig. 11.2 is not tangent to the y axis at the origin. The tangent at the origin and the tangent at (7, 5) can be constructed geometrically by the construction described in Art. 10.5, or the slope of the tangents can be calculated analytically from the derivative, which in this example is

$$\frac{dy}{dx} = 2 - \frac{6x}{2\sqrt{3x^2 - 3x + 6.25}}$$

In this example, when $x = 0$ then $dy/dx = 2.6$, and so the slope

of the tangent at the origin is 2.6, *i.e.*, the trigonometric tangent of the angle between the tangent line and the x axis is 2.6 at (0, 0). Also, when $x = 7$, $dy/dx = 0.30435$, and so the slope of the tangent at (7, 5) is 0.30435, *i.e.*, the trigonometric tangent of the angle between the tangent line and the x axis is 0.30435 at (7, 5).

11.3. Conic determined by a point slope and three points. Having established the theory for the case of a conic through five points, we shall now specialize the theory to certain special cases. Consider the case of a conic determined by a point slope and

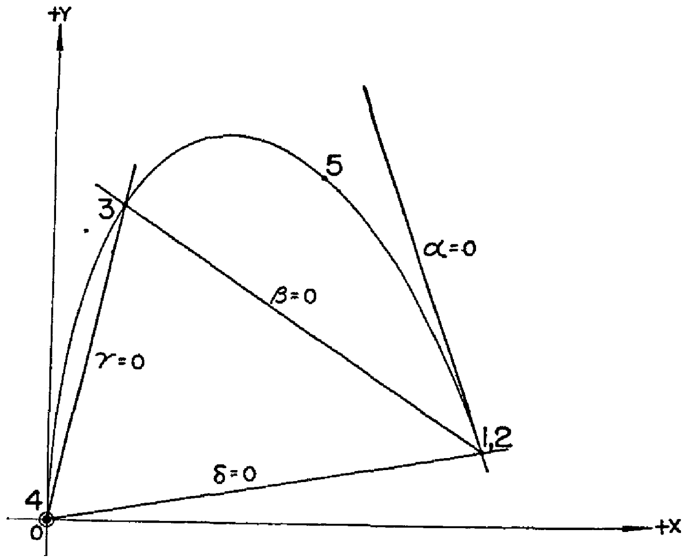


FIG. 11.3.

three points. By a point slope we mean that a point is given and the slope of the curve at that point is also given. Suppose that in Fig. 11.1 the points 1 and 2 coincide. The line through 1 and 2 becomes a tangent to the curve (see Fig. 11.3). There are five given conditions, namely, four points and a tangent. The procedure to be used in arriving at the equation of the curve is the same as in Art. 11.2.

Example. Find the equation of the conic which is determined by the points 4(0, 0), 3(1, 2), and 5(2, 3), and the tangent at the point 1, 2 (4, 4), the equation of the tangent being $y - \frac{1}{4}x - 3 = 0$.

Referring to Fig. 11.3, the three given points are 4, 3, 5, and the tangent is at the point 1, 2. Here 1, 2 indicates that the points 1 and 2 (of the general case of a conic through five points) coincide, and the line through 1, 2 is tangent to the curve.

The basic straight line equations are

$$\begin{aligned}\alpha : y - \tfrac{1}{4}x - 3 &= 0,\\ \beta : y - \tfrac{2}{3}x - \tfrac{4}{3} &= 0,\\ \gamma : x - \tfrac{1}{2}y &= 0,\\ \delta : y - x &= 0.\end{aligned}$$

Comparing these equations with the general equations for α, β, γ, δ in Art. 11.2, we find

$$\begin{aligned}m_1 &= \tfrac{1}{4}.\\ m_2 &= \tfrac{2}{3}.\\ m_3 &= \tfrac{1}{2}.\\ m_4 &= 1.\\ h_1 &= 3.\\ h_2 &= \tfrac{4}{3}.\end{aligned}$$

The value of K is calculated from the relation $K = -\dfrac{\alpha\gamma}{\beta\delta}$, evaluated for $x = 2$, $y = 3$. The result is $K = 0.75$.

Calculate the values of A, B, D, E, F from the general formulas in Art. 11.2. The results are $A = 0.25$, $B = -0.125$, $D = 0.25$, $E = 0.5$, $F = -2$. Calculate the values of a, b, c, d from the general formulas in Art. 11.2. The results are $c = 0.25$, $d = -1$, $a = -0.9375$, $b = 7.5$. The final equation of the curve is therefore

$$y = 0.25x - 1 + \sqrt{-0.9375x^2 + 7.5x + 1}.$$

The plus sign for the radical is selected by substituting the x coordinate of one of the given points and choosing the sign of the radical which will check the y coordinate of that point.

After obtaining the equation of a conic, it is desirable to check the given data and thereby prove that the equation satisfies them. In this example it is necessary to check the points (0, 0), (1, 2), (2, 3), and (4, 4). It is also necessary to check the slope of the given tangent at the point (4, 4), which depends upon the value of the derivative

$$\frac{dy}{dx} = 0.25 + \frac{-1.8750x + 7.5}{2\sqrt{-0.9375x^2 + 7.5x + 1}}$$

at the point (4, 4). At (4, 4) $dy/dx = 0.25$, which checks with the slope of the given tangent line $y - \frac{1}{4}x - 3 = 0$.

11.4. Conic determined by one point and two point slopes. Consider the problem of finding the equation of a conic determined by one point and two point slopes (see Fig. 11.4). This is the most useful of all the cases we consider. Very often, in design and in lofting, it is necessary to establish a curve that is tangent to a given line at a given point, is tangent to another

given line at a given point, and passes through a third given point. Conic sections as calculated in this article satisfy these five conditions perfectly. The third given point (control point) allows great latitude in fixing the shape of the curve after the two point slopes have been fixed. This flexibility is very valuable and is one of the chief reasons why conic sections are so useful in designing and lofting.

Compare Figs. 11.4 and 11.1. The points 1, 2 of Fig. 11.1 coincide in Fig. 11.4, and the points 3, 4 of Fig. 11.1 coincide

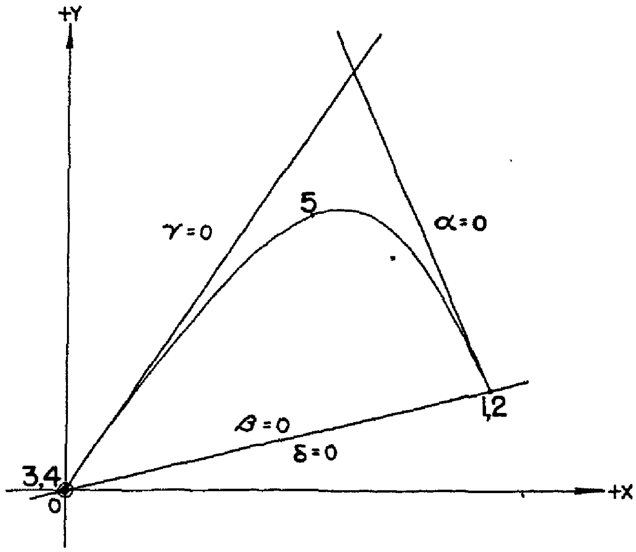


FIG. 11.4.

in Fig. 11.4. The line $\beta = 0$, which joins points 2, 3 in Fig. 11.1, and the line $\delta = 0$, which joins points 4, 1 in Fig. 11.1, therefore coincide in Fig. 11.4. The lines 1-2 and 3-4 of Fig. 11.1 become tangents to the conic in Fig. 11.4.

Example 1. Find the equation of the conic that is tangent to the line $y = \frac{5}{3}x$ at the origin, is tangent to the line through the two points (3, 5) and (4, 4) at the point (4, 4), and passes through the point (3.5, 4). See Fig. 11.5.

Notice that this type of problem is a special case of the conic through five points (see Fig. 11.1). The points 4, 3 coincide at the origin in Fig. 11.5, the points 2, 1 coincide at (4, 4), and the point 5 is the point (3.5, 4).

The equation of the family of conics tangent to $y = \frac{5}{3}x$ at the origin and tangent to the line through (3, 5) and (4, 4) at the point (4, 4) is

$$K\beta\delta + \alpha\gamma = 0,$$

where the general equations of α, β, γ, δ are as given in Art. 11.2.

In this example, the equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ become

$$\alpha: y + x - 8 = 0,$$
$$\beta: y - x = 0,$$
$$\gamma: x - 0.6y = 0,$$
$$\delta: y - x = 0.$$

By comparing these equations with the general equations in Art. 11.2,

$$m_1 = -1. \qquad h_1 = 8.$$
$$m_2 = 1. \qquad h_2 = 0.$$
$$m_3 = 0.6.$$
$$m_4 = 1.$$

Also, $K = -\dfrac{\alpha\gamma}{\beta\delta}$, evaluated for $x = 3.5$ and $y = 4$; so $K = 2.2$. Using

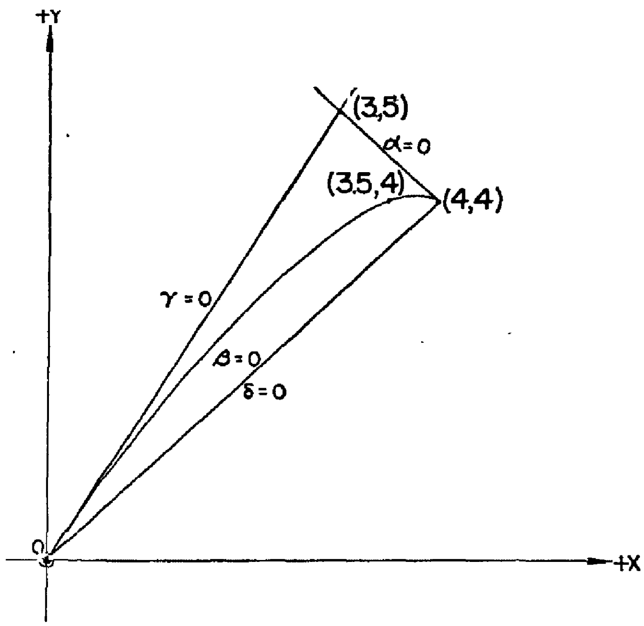


FIG. 11.5.

these values to calculate A, B, D, E, and F,

$$A = 1.6,$$
$$B = -4,$$
$$D = 3.2,$$
$$E = 4.8,$$
$$F = -8.$$

Calculating the values of a, b, c, and d from these numbers, by means of the general formulas given in Art. 11.2,

$$c = 1.25,$$
$$d = -1.5,$$
$$a = -0.4375,$$
$$b = 1.25.$$

Therefore the final equation of the curve is

$$y = 1.25x - 1.5 + \sqrt{-0.4375x^2 + 1.25x + 2.25}.$$

It is always advisable to check the given data. In this case, we check the points (0, 0), (4, 4), and (3.5, 4). Also, we check the slope at (0, 0) and (4, 4) by means of the derivative

$$\frac{dy}{dx} = 1.25 + \frac{-0.8750x + 1.25}{2\sqrt{-0.4375x^2 + 1.25x + 2.25}}.$$

Notice that the formulas given in Art. 11.2 are quite general and include this very useful case of a conic determined by one point and two point slopes.

Example 2. Determine the equation of the nose of the fuselage from the data given in Fig. 11.6. Let the control point be (15, 20).

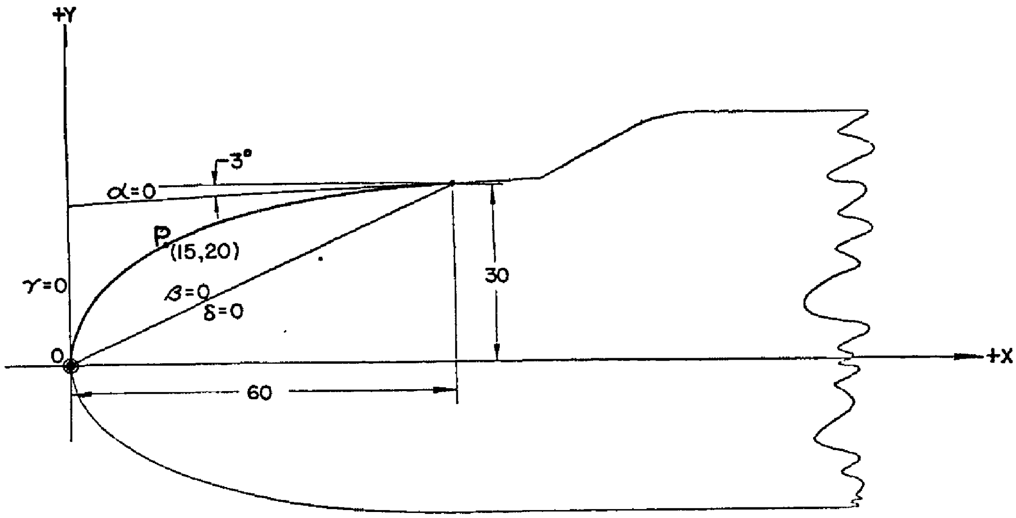


FIG. 11.6.

The steps in the calculation of the constants are the same as in Example 1. The results are

$$\alpha: y - 0.05241x - 26.8554 = 0.$$
$$\beta: y - 0.5x = 0.$$
$$\gamma: x = 0.$$
$$\delta: y - 0.5x = 0.$$

$K = 0.73359.$	$A = 0.73359.$	$a = -0.14559.$
$m_1 = 0.05241.$	$B = 0.26641.$	$b = 36.608.$
$m_2 = 0.5.$	$D = 0.13099.$	$c = -0.18158.$
$m_3 = 0.$	$E = 0.$	$d = 0.$
$m_4 = 0.5.$	$F = -26.8554.$	
$h_1 = 26.8554.$		
$h_2 = 0$		

$$y = -0.18158x + \sqrt{-0.14559x^2 + 36.608x}.$$

It is essential to check the given data. From this equation the curve can be plotted by ordinates. The control point can be varied to produce an infinite variety of shapes.

11.5. The control point. The control point can be varied at will. Each position of the control point will result in a unique value of K, and therefore in a unique conic. A convenient method of selecting the control point is illustrated in Fig. 11.7.

In Fig. 11.7, the point slopes at A and B are fixed. The point M is the mid-point of the line segment AB. The point E is

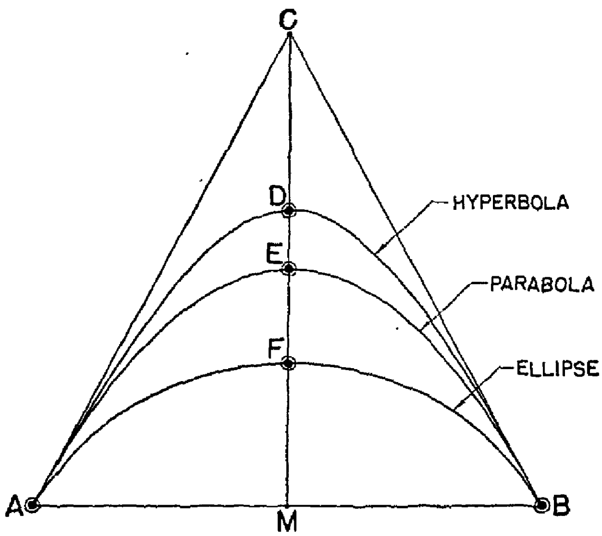


FIG. 11.7.

the mid-point of the line segment CM. The control point is established on CM. The figure shows three positions for the control point: D, E, and F. The position at D yields a hyperbola, the position at E yields a parabola, and the position at F yields an ellipse. Let us define the ratio MP/CM as the ρ value of the conic, where P represents the position of the control point. It can be proved that the nature of the conic is determined as follows:

$\rho < \frac{1}{2}$.	ellipse
$\rho = \frac{1}{2}$.	parabola
$\rho > \frac{1}{2}$.	hyperbola

With fixed point slopes at A and B it is therefore convenient to select a control point on CM and to designate the curve by the value of ρ. For example, $\rho = 0.25$ indicates a unique conic,

which happens to be an ellipse. Notice that only one value of ρ yields a parabola, namely, $\rho = \frac{1}{2}$.

11.6. The parabola. Consider the problem of determining a conic that is tangent to a given line at a given point, and is tangent to another given line at a given point. If these are the only restrictions on the curve, the control point is of no particular significance. A convenient control point for such a curve is one that yields a parabola. In this case the equation

$$y = cx + d \pm \sqrt{ax^2 + bx + d^2}\,.$$

reduces to the special form

$$y = m\sqrt{x} + nx,$$

if the axes are taken as in Fig. 11.8. With the dimensions given in Fig. 11.8, the values of m and n are

$$m = \frac{2t}{\sqrt{r}},$$

$$n = s - 2t$$

This equation for a parabola can be derived as follows: From Figs. 11.8 and 11.4,

$$\alpha : y - \frac{s-t}{r}x - t,$$

$$\beta : y - \frac{s}{r}x,$$

$$\gamma : x,$$

$$\delta : y - \frac{s}{r}x.$$

The equation of the family of conics with the point slopes as given in Fig. 11.8 is

$$K\left(y - \frac{s}{r}x\right)\left(y - \frac{s}{r}x\right) + \left(y - \frac{s-t}{r}x - t\right)(x) = 0.$$

$$\left(\frac{Ks^2}{r^2} - \frac{s-t}{r}\right)x^2 + \left(\frac{-2sK}{r} + 1\right)xy + Ky^2 - tx = 0.$$

The conic will be a parabola if $B^2 - 4AD = 0$, where A is the coefficient of x^2, B is the coefficient of xy, and D is the coefficient of y^2. This condition yields

$$K = \frac{r}{4t}.$$

Substituting this value for K in the equation of the family of

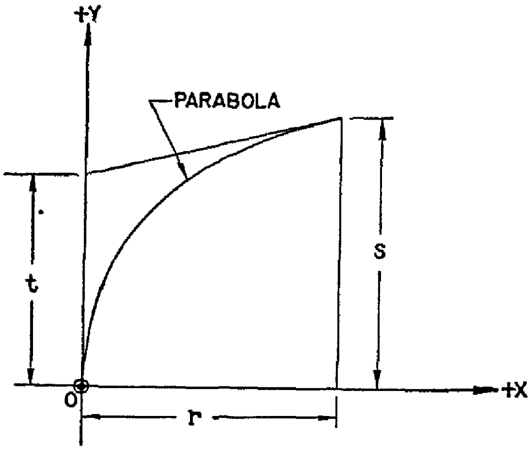


FIG. 11.8.

conics, and solving for y in terms of x,

$$y = \frac{2t}{\sqrt{r}} \sqrt{x} + \frac{s - 2t}{r} x.$$

This completes the derivation.

Example. Find the equation of the parabola as determined in Fig. 11.

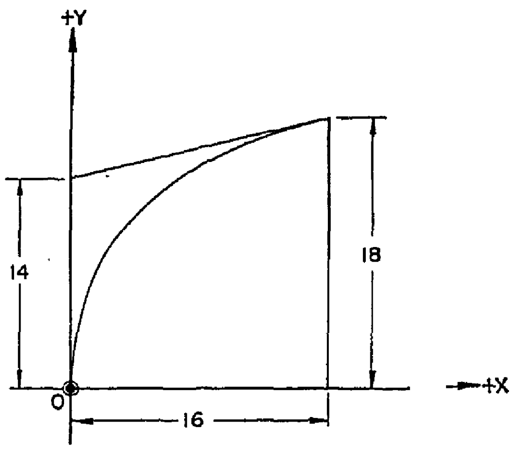


FIG. 11.9.

Referring to Fig. 11.8 and the formulas for m and n,

$$r = 16,$$
$$s = 18,$$
$$t = 14,$$
$$\frac{2t}{\sqrt{r}} = \ ,$$
$$\frac{s - 2t}{\ } = -0.625.$$

The equation of the parabola is

$$y = 7\sqrt{x} - 0.625x.$$

To check the given data, check the points (0, 0) and (16, 18) and check the slope at these two points by the formula

$$\frac{dy}{dx} = \frac{m}{2\sqrt{x}} + n.$$

If the axes of reference are such that it is not convenient to use one of the tangents as the y axis, as was done in Fig. 11.9, it is best to use the general method as outlined in Art. 11.4.

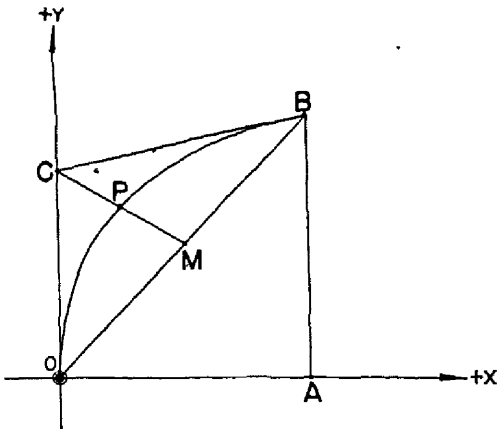


FIG. 11.10.

For parabolas of the type shown in Figs. 11.8 and 11.9, the control point is as shown in Fig. 11.10.

In Fig. 11.10, M is the mid-point of OB and P is the mid-point of CM. The point P is the control point. Notice that

$$PM/CM = \rho = \tfrac{1}{2}.$$

11.7. Matching conics. Two circles are considered to be "matched" when they have a common tangent (see Fig. 11.11). In Fig. 11.11, the common tangent is the y axis, and the line of centers is the x axis. Although the tangents are identical at the origin, there is an abrupt change in the curvature at that point. The curvature of a circle is the reciprocal of its radius, and, since the radii of the two circles are different, their curvatures are different. We use the word *curvature* here to mean the rate of change of the direction of the tangent, as defined precisely in calculus.

If we replace one of the circles by a straight line, we have the situation shown in Fig. 11.12. This is similar to the railroad engineering problem of joining a straight track to a circular

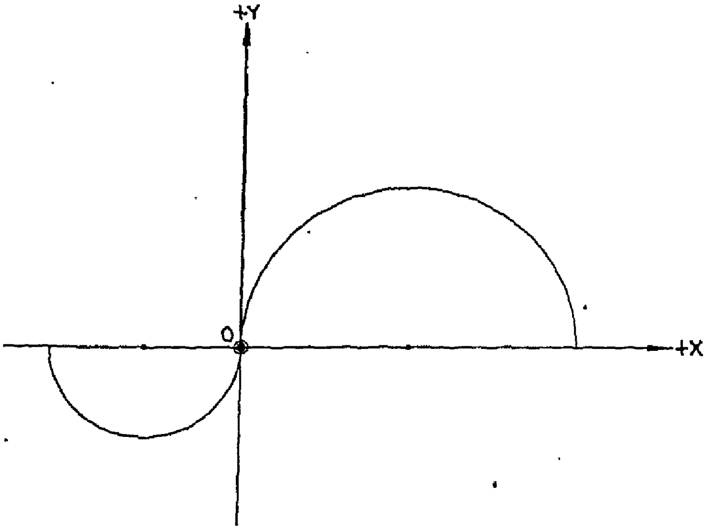


FIG. 11.11.

track. An object moving along the straight line will acquire an acceleration as it meets the circular arc. In railroad engineering certain transition curves are used to make the change in curvature from the straight line to the circle as gradual as possible.

A somewhat similar situation arises when we attempt to match two conics (see Fig. 11.13).

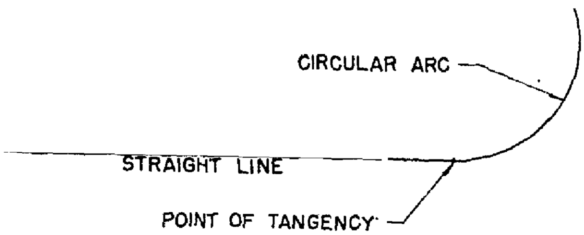


FIG. 11.12.

In Fig. 11.13, conic 1 is to be matched to conic 2. Usually this is accomplished by merely establishing the y axis as the common tangent. That this is not sufficient to produce a matching of curvatures results from the fact that the curvature of a conic varies from point to point on a conic. That the matching of curvatures is desirable follows from the discussion of the preceding paragraphs. Aerodynamic considerations demand that

the curvatures of conics 1 and 2 be the same at their common point, at least for certain critical places on the contour of the airplane.

One method of stating the problem is as follows: Let us assume that conic 1 in Fig. 11.13 is a given conic, determined by two point slopes and a control point. It is required to find the equation of a conic 2 that will

1. Be tangent to conic 1 at the origin.
2. Have the same curvature at the origin as conic 1.

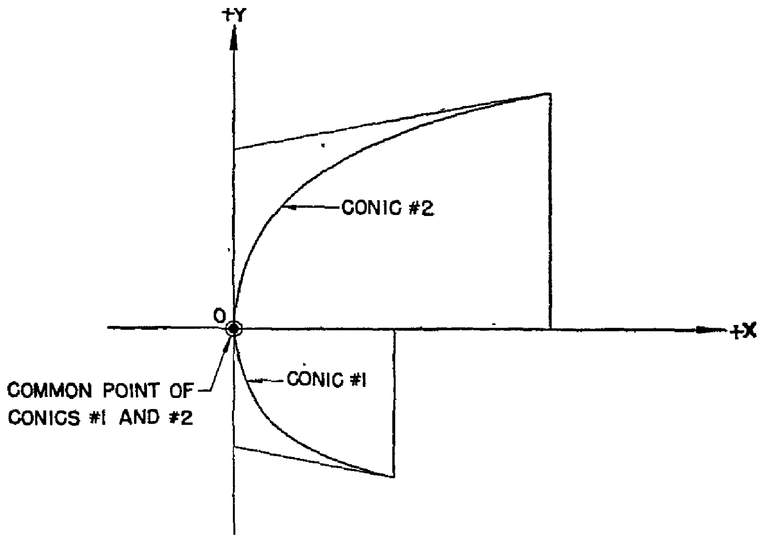


FIG. 11.13.

3. Be tangent to a given line at a given point. Notice that an arbitrary control point cannot be assigned on conic 2. Such a control point would result in six conditions on conic 2. Only five conditions can be met by a conic. One method of solving this problem is illustrated in the following example.

Example. Find the equation of the conic as determined in Fig. 11.14. The radius of the circle of curvature is equal to the reciprocal of the curvature. The equation of the circle of curvature at the origin is

$$x^2 + y^2 - \tfrac{2}{9}x = 0.$$

Assume an equation of the type

$$Ax^2 + Bxy + Cy^2 + Dx = 0.$$

Ordinarily a conic and a circle will intersect in four points. In the case of the circle of curvature, three of these points coincide, in this example, at the

origin. We have the problem of solving the above two equations so that

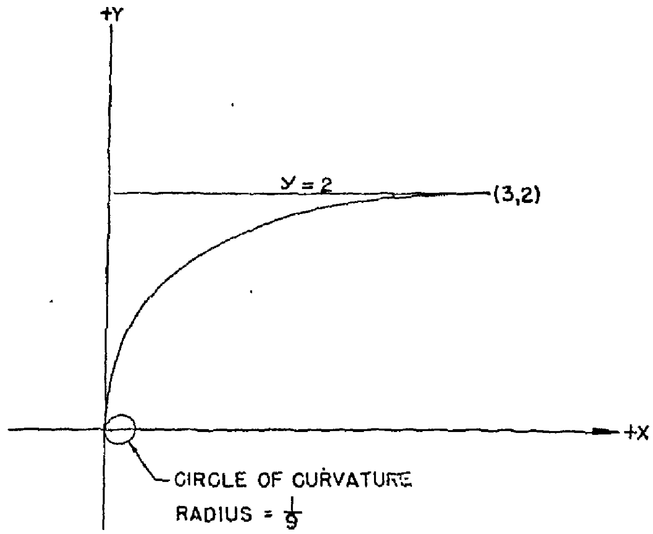


FIG. 11.14.

three of the roots will be equal. In this case,

$$= -\frac{2}{},$$

so we have

$$Ax^2 + Bxy + 9y^2 - 2x = 0.$$

Let us make the conic pass through the point (3, 2). We obtain

$$3A + 2B = -10.$$

Now differentiate the equation of the conic implicitly:

$$\frac{dy}{dx} = \frac{-2Ax - By + 2}{Bx + 18y}.$$

At the point (3, 2) the tangent is horizontal in Fig. 11.14. Therefore

$$-6A - 2B + 2 = 0.$$

Solving for A and B,

$$A = 4, \qquad B = -11.$$

The equation of the required conic is

$$4x^2 - 11xy + 9y^2 - 2x = 0.$$

This conic will satisfy the following five conditions:

1. Pass through the origin.
2. Have a vertical tangent at the origin.
3. Have a curvature of 9 at the origin.
4. Pass through the point (3, 2).
5. Have a slope of zero at (3, 2).

APPENDIX

SUMMARY OF FORMULAS

I. PLANE ANALYTIC GEOMETRY

1. Length of the line segment $P_1(x_1, y_1)P_2(x_2, y_2)$.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

2. Inclination of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

$$\alpha = \tan^{-1}\frac{y_2 - y_1}{x_2 - x_1}.$$

3. Slope of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

4. Point-slope equation of a straight line.

$$y - y_1 = m(x - x_1).$$

5. Two-point equation of a straight line.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

6. Slope-intercept equation of a straight line.

$$y = mx + b.$$

7. Distance from the point $P_1(x_1, y_1)$ to the line $y = mx + b$.

$$d = \frac{mx_1 - y_1 + b}{\sqrt{m^2 + 1}}$$

II. SOLID ANALYTIC GEOMETRY

8. Direction ratios of the line through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

$$x_2 - x_1 : y_2 - y_1 : z_2 - z_1.$$

9. Length of the line segment $P_1(x_1, y_1, z_1)P_2(x_2, y_2, z_2)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

10. Direction cosines of the line through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

$$\cos\alpha = \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}.$$

$$\cos\beta = \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}.$$

$$\cos\gamma = \frac{z_2 - z_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}.$$

11. True angle between two lines whose direction cosines are a, b, c and d, e, f.

$$\cos\theta = ad + be + cf.$$

12. True angle between two lines whose direction ratios are $r:s:t$ and $u:v:w$.

$$\cos\theta = \frac{ru + sv + tw}{\sqrt{r^2 + s^2 + t^2}\sqrt{u^2 + v^2 + w^2}}.$$

13. True angle (θ) between the line whose direction cosines are a, b, c and the plane, the direction cosines of a normal to the plane being d, e, f.

$$\cos(90^\circ - \theta) = ad + be + cf.$$

14. True angle between two planes, the direction cosines of normals to the two planes being a, b, c and d, e, f.

$$\cos\theta = ad + be + cf.$$

15. Direction ratios of a normal to a plane, the direction ratios of two lines in the plane being a, b, c and d, e, f.

$$bf - ce : cd - af : ae - bd.$$

16. Normal form of the equation of a plane, the direction cosines of a normal to the plane being $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and the perpendicular distance from the origin to the plane being p.

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p.$$

17. General form of the equation of a plane.

$$Ax + By + Cz + D = 0.$$

18. Equation of the plane determined by the point $P_1(x_1, y_1, z_1)$ and a normal to the plane, the direction ratios of the normal being $a:b:c$.

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

19. Distance from the point $P_1(x_1, y_1, z_1)$ to the plane

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p.$$

$$d = x_1\cos\alpha + y_1\cos\beta + z_1\cos\gamma - p.$$

20. Test for parallel planes.

$$A_1x + B_1y + C_1z + D_1 = 0.$$
$$A_2x + B_2y + C_2z + D_2 = 0.$$
$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

21. Test for perpendicular planes.

$$A_1x + B_1y + C_1z + D_1 = 0.$$
$$A_2x + B_2y + C_2z + D_2 = 0.$$
$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

22. Intercept form of the equation of a plane whose intercepts are a, b, c.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

23. Equations of a line. General form.

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0.$$

24. Equations of a line. Projecting planes.

$$A_1x + B_1y = D_1,$$
$$A_2x + C_2z = D_2.$$

25. Equations of a line through the point $P_1(x_1, y_1, z_1)$, and whose direction ratios are $a:b:c$.

$$x_1 \qquad y - y_1 \qquad z - z_1$$

26. Distance from the point $P_1(x_1, y_1, z_1)$ to the line $P_2(x_2, y_2, z_2)P_3(x_3, y_3, z_3)$, where θ is the true angle between P_1P_2 and P_2P_3.

$$d = P_1P_2 \sin \theta.$$

27. Shortest distance between the lines AB and CD, where $P_1(x_1, y_1, z_1)$ is any point on AB, $P_2(x_2, y_2, z_2)$ is any point on CD, and $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of a normal to AB and CD.

$$d = (x_1 - x_2) \cos \alpha + (y_1 - y_2) \cos \beta + (z_1 - z_2) \cos \gamma.$$

28. Direction ratios of the bisector of the angle between two lines whose direction cosines are $a:b:c$ and $d:e:f$.

$$a + d:b + e:c + f.$$

29. Equations for translating axes.

$$x' = x - h.$$
$$y' = y - k.$$
$$z' = z - l.$$

30. Rotation of axes equations. Rigged to wing reference plane.

$$x_w = x \cos \phi + z \sin \phi.$$
$$y_w = y.$$
$$z_w = -x \sin \phi + z \cos \phi.$$

31. Rotation of axes equations. Wing reference plane to rigged.

$$x = x_w \cos \phi - z_w \sin \phi.$$
$$y = y_w.$$
$$z = x_w \sin \phi + z_w \cos \phi.$$

32. Rotation of axes equations. Rigged to vertical stabilizer.

$$x_v = x \cos \delta + y \sin \delta.$$
$$y_v = -x \sin \delta + y \cos \delta.$$
$$z_v = z.$$

33. Rotation of axes equations. Vertical stabilizer to rigged.

$$x = x_v \cos \delta - y_v \sin \delta.$$
$$y = x_v \sin \delta + y_v \cos \delta.$$
$$z = z_v.$$

34. Rotation of axes equations. Rigged to nacelle.

$$x_n = x.$$
$$y_n = y \cos \delta - z \sin \delta.$$
$$z_n = y \sin \delta + z \cos \delta.$$

35. Rotation of axes equations. Nacelle to rigged.

$$x = x_n.$$
$$y = y_n \cos \delta + z_n \sin \delta.$$
$$z = -y_n \sin \delta + z_n \cos \delta.$$

36. Rotation of axes equations. Rigged to horizontal stabilizer.

$$x_h = x.$$
$$y_h = y \cos \theta - z \sin \theta.$$
$$z_h = y \sin \theta + z \cos \theta.$$

37. Rotation of axes equations. Horizontal stabilizer to rigged.

$$x = x_h.$$
$$y = y_h \cos \theta + z_h \sin \theta.$$
$$z = -y_h \sin \theta + z_h \cos \theta.$$

38. Rotation of axes equations. Rigged to wing chord plane.

$$x_w = x \cos \phi + y \sin \phi \sin \theta + z \sin \phi \cos \theta.$$
$$y_w = y \cos \theta - z \sin \theta.$$
$$z_w = -x \sin \phi + y \cos \phi \sin \theta + z \cos \phi \cos \theta.$$

39. Rotation of axes equation. Wing chord plane to rigged.

$$
\begin{aligned}
x &= x_w \cos \phi - z_w \sin \phi. \\
y &= x_w \sin \phi \sin \theta + y_w \cos \theta + z_w \cos \phi \sin \theta. \\
z &= x_w \sin \phi \cos \theta - y_w \sin \theta + z_u \cos \phi \cos \theta.
\end{aligned}
$$

40. General equation of a conic.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

	0°		1°		2°		3°		
′	tan	cotan	tan	cotan	tan	cotan	tan	cotan	′
0	.00000	Infinite.	.01746	57.2900	.03492	28.6363	.05241	19.0811	60
1	.00029	3437.750	.01775	56.3506	.03521	28.3994	.05270	18.9755	59
2	.00058	1718.870	.01804	55.4415	.03550	28.1664	.05299	18.8711	58
3	.00087	1145.920	.01833	54.5613	.03579	27.9372	.05328	18.7678	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	33
28	.00814	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	15
46	.01338	74.7292	.03084	32.4213	.04832	20.6932	.06584	15.1893	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	0
′	cotan	tan	cotan	tan	cotan	tan	cotan	tan	′
	89°		88°		87°		86°		

	4°		5°		6°		7°		
′	tan	cotan	tan	cotan	tan	cotan	tan	cotan	′
0′	.06993	14.3007	.08749	11.4301	.10510	9.51436	.12278	8.14435	60
1	.07022	14.2411	.08778	11.3919	.10540	9.48781	.12308	8.12481	59
2	.07051	14.1821	.08807	11.3540	.10569	9.46141	.12338	8.10536	58
3	.07080	14.1235	.08837	11.3163	.10599	9.43515	.12367	8.08600	57
4	.07110	14.0655	.08866	11.2789	.10628	9.40904	.12397	8.06674	56
5	.07139	14.0079	.08895	11.2417	.10657	9.38307	.12426	8.04756	55
6	.07168	13.9507	.08925	11.2048	.10687	9.35724	.12456	8.02848	54
7	.07197	13.8940	.08954	11.1681	.10716	9.33154	.12485	8.00948	53
8	.07227	13.8378	.08983	11.1316	.10746	9.30599	.12515	7.99058	52
9	.07256	13.7821	.09013	11.0954	.10775	9.28058	.12544	7.97176	51
10	.07285	13.7267	.09042	11.0594	.10805	9.25530	.12574	7.95302	50
11	.07314	13.6719	.09071	11.0237	.10834	9.23016	.12603	7.93438	49
12	.07344	13.6174	.09101	10.9882	.10863	9.20516	.12633	7.91582	48
13	.07373	13.5634	.09130	19.9529	.10893	9.18028	.12662	7.89734	47
14	.07402	13.5098	.09159	10.9178	.10922	9.15554	.12692	7.87895	46
15	.07431	13.4566	.09189	10.8829	.10952	9.13093	.12722	7.86064	45
16	.07461	13.4039	.09218	10.8483	.10981	9.10646	.12751	7.84242	44
17	.07490	13.3515	.09247	10.8139	.11011	9.08211	.12781	7.82428	43
18	.07519	13.2996	.09277	10.7797	.11040	9.05789	.12810	7.80622	42
19	.07548	13.2480	.09306	10.7457	.11070	9.03379	.12840	7.78825	41
20	.07578	13.1969	.09335	10.7119	.11099	9.00983	.12869	7.77035	40
21	.07607	13.1461	.09365	10.6783	.11128	8.98598	.12899	7.75254	39
22	.07636	13.0958	.09394	16.6450	.11158	8.96227	.12929	7.73480	38
23	.07665	13.0458	.09423	10.6118	.11187	8.93867	.12958	7.71715	37
24	.07695	12.9962	.09453	10.5789	.11217	8.91520	.12988	7.69957	36
25	.07724	12.9469	.09482	10.5462	.11246	8.89185	.13017	7.68208	35
26	.07753	12.8981	.09511	10.5136	.11276	8.86862	.13047	7.66466	34
27	.07782	12.8496	.09541	10.4813	.11305	8.84551	.13076	7.64732	33
28	.07812	12.8014	.09570	10.4491	.11335	8.82252	.13106	7.63005	32
29	.07841	12.7536	.09600	10.4172	.11364	8.79964	.13136	7.61287	31
30	.07870	12.7062	.09629	10.3854	.11394	8.77689	.13165	7.59575	30
31	.07899	12.6591	.09658	10.3538	.11423	8.75425	.13195	7.57872	29
32	.07929	12.6124	.09688	10.3224	.11452	8.73172	.13224	7.56176	28
33	.07958	12.5660	.09717	10.2913	.11482	8.70931	.13254	7.54487	27
34	.07987	12.5199	.09746	10.2602	.11511	8.68701	.13284	7.52806	26
35	.08017	12.4742	.09776	10.2294	.11541	8.66482	.13313	7.51132	25
36	.08046	12.4288	.09805	10.1988	.11570	8.64275	.13343	7.49465	24
37	.08075	12.3838	.09834	10.1683	.11600	8.62078	.13372	7.47806	23
38	.08104	12.3390	.09864	10.1381	.11629	8.59893	.13402	7.46154	22
39	.08134	12.2946	.09893	10.1080	.11659	8.57718	.13432	7.44509	21
40	.08163	12.2505	.09923	10.0780	.11688	8.55555	.13461	7.42871	20
41	.08192	12.2067	.09952	10.0483	.11718	8.53402	.13491	7.41240	19
42	.08221	12.1632	.09981	10.0187	.11747	8.51259	.13521	7.39616	18
43	.08251	12.1201	.10011	9.98931	.11777	8.49128	.13550	7.37999	17
44	.08280	12.0772	.10040	9.96007	.11806	8.47007	.13580	7.36389	16
45	.08309	12.0346	.10069	9.93101	.11836	8.44896	.13609	7.34786	15
46	.08339	11.9923	.10099	9.90211	.11865	8.42795	.13639	7.33190	14
47	.08368	11.9504	.10128	9.87338	.11895	8.40705	.13669	7.31600	13
48	.08397	11.9087	.10158	9.84482	.11924	8.38625	.13698	7.30018	12
49	.08427	11.8673	.10187	9.81641	.11954	8.36555	.13728	7.28442	11
50	.08456	11.8262	.10216	9.78817	.11983	8.34496	.13758	7.26873	10
51	.08485	11.7853	.10246	9.76009	.12013	8.32446	.13787	7.25310	9
52	.08514	11.7448	.10275	9.73217	.12042	8.30406	.13817	7.23754	8
53	.08544	11.7045	.10305	9.70441	.12072	8.28376	.13846	7.22204	7
54	.08573	11.6645	.10334	9.67680	.12101	8.26355	.13876	7.20661	6
55	.08602	11.6248	.10363	9.64935	.12131	8,24345	.13906	7.19125	5
56	.08632	11.5853	.10393	9.62205	.12160	8.22344	.13935	7.17594	4
57	.08661	11.5461	.10422	9.59490	.12190	8.20352	.13965	7.16071	3
58	.08690	11.5072	.10452	9.56791	.12219	8.18370	.13995	7.14553	2
59	.08720	11.4685	.10481	9.54106	.12249	8.16398	.14024	7.13042	1
60	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	0
′	cotan	tan	cotan	tan	cotan	tan	cotan	tan	′
	85°		84°		83°		82°		

′	8° tan	8° cotan	9° tan	9° cotan	10° tan	10° cotan	11° tan	11° cotan	′
0	.14054	7.11537	.15838	6.31375	.17633	5.67128	.19438	5.14455	60
1	.14084	7.10038	.15868	6.30189	.17663	5.66165	.19468	5.13658	59
2	.14113	7.08546	.15898	6.29007	.17693	5.65205	.19498	5.12862	58
3	.14143	7.07059	.15928	6.27829	.17723	5.64248	.19529	5.12069	57
4	.14173	7.05579	.15958	6.26655	.17753	5.63295	.19559	5.11279	56
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	55
6	.14232	7.02637	.16017	6.24321	.17813	5.61397	.19619	5.09704	54
7	.14262	7.01174	.16047	6.23160	.17843	5.60452	.19649	5.08921	53
8	.14291	6.99718	.16077	6.22003	.17873	5.59511	.19680	5.08139	52
9	.14321	6.98268	.16107	6.20851	.17903	5.58573	.19710	5.07360	51
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	50
11	.14381	6.95385	.16167	6.18559	.17963	5.56706	.19770	5.05809	49
12	.14410	6.93952	.16196	6.17419	.17993	5.55777	.19801	5.05037	48
13	.14440	6.92525	.16226	6.16283	.18023	5.54851	.19831	5.04267	47
14	.14470	6.91104	.16256	6.15151	.18053	5.53927	.19861	5.03499	46
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	45
16	.14529	6.88278	.16316	6.12899	.18113	5.52090	.19921	5.01971	44
17	.14559	6.86874	.16346	6.11779	.18143	5.51176	.19952	5.01210	43
18	.14588	6.85475	.16376	6.10664	.18173	5.50264	.19982	5.00451	42
19	.14618	6.84082	.16405	6.09552	.18203	5.49356	.20012	4.99695	41
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	40
21	.14678	6.81312	.16465	6.07340	.18263	5.47548	.20073	4.98188	39
22	.14707	6.79936	.16495	6.06240	.18293	5.46648	.20103	4.97438	38
23	.14737	6.78564	.16525	6.05143	.18323	5.45751	.20133	4.96690	37
24	.14767	6.77199	.16555	6.04051	.18353	5.44857	.20164	4.95945	36
25	.14796	6.75838	.16585	6.02962	.18383	5.43966	.20194	4.95201	35
26	.14826	6.74483	.16615	6.01878	.18414	5.43077	.20224	4.94460	34
27	.14856	6.73133	.16645	6.00797	.18444	5.42192	.20254	4.93721	33
28	.14886	6.71789	.16674	5.99720	.18474	5.41309	.20285	4.92984	32
29	.14915	6.70450	.16704	5.98646	.18504	5.40429	.20315	4.92249	31
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	30
31	.14975	6.67787	.16764	5.96510	.18564	5.38677	.20376	4.90785	29
32	.15005	6.66463	.16794	5.95448	.18594	5.37805	.20406	4.90056	28
33	.15034	6.65144	.16824	5.94390	.18624	5.36936	.20436	4.89330	27
34	.15064	6.63831	.16854	5.93335	.18654	5.36070	.20466	4.88605	26
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87882	25
36	.15124	6.61219	.16914	5.91235	.18714	5.34345	.20527	4.87162	24
37	.15153	6.59921	.16944	5.90191	.18745	5.33487	.20557	4.86444	23
38	.15183	6.58627	.16974	5.89151	.18775	5.32631	.20588	4.85727	22
39	.15213	6.57339	.17004	5.88114	.18805	5.31778	.20618	4.85013	21
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	20
41	.15272	6.54777	.17063	5.86051	.18865	5.30080	.20679	4.83590	19
42	.15302	6.53503	.17093	5.85024	.18895	5.29235	.20709	4.82882	18
43	.15332	6.52234	.17123	5.84001	.18925	5.28393	.20739	4.82175	17
44	.15362	6.50970	.17153	5.82982	.18955	5.27553	.20770	4.81471	16
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	15
46	.15421	6.48456	.17213	5.80953	.19016	5.25880	.20830	4.80068	14
47	.15451	6.47206	.17243	5.79944	.19046	5.25048	.20861	4.79370	13
48	.15481	6.45961	.17273	5.78938	.19076	5.24218	.20891	4.78673	12
49	.15511	6.44720	.17303	5.77936	.19106	5.23391	.20921	4.77978	11
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	10
51	.15570	6.42253	.17363	5.75941	.19166	5.21744	.20982	4.76595	9
52	.15600	6.41026	.17393	5.74949	.19197	5.20925	.21013	4.75906	8
53	.15630	6.39804	.17423	5.73960	.19227	5.20107	.21043	4.75219	7
54	.15660	6.38587	.17453	5.72974	.19257	5.19293	.21073	4.74534	6
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	5
56	.15719	6.36165	.17513	5.71013	.19317	5.17671	.21134	4.73170	4
57	.15749	6.34961	.17543	5.70037	.19347	5.16863	.21164	4.72490	3
58	.15779	6.33761	.17573	5.69064	.19378	5.16058	.21195	4.71813	2
59	.15809	6.32566	.17603	5.68094	.19408	5.15256	.21225	4.71137	1
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	0
′	81° cotan	81° tan	80° cotan	80° tan	79° cotan	79° tan	78° cotan	78° tan	′

′	12° tan	12° cotan	13° tan	13° cotan	14° tan	14° cotan	15° tan	15° cotan	′
0	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	60
1	.21286	4.69791	.23117	4.32573	.24964	4.00582	.26826	3.72771	59
2	.21316	4.69121	.23148	4.32001	.24995	4.00086	.26857	3.72338	58
3	.21347	4.68452	.23179	4.31430	.25026	3.99592	.26888	3.71907	57
4	.21377	4.67786	.23209	4.30860	.25056	3.99099	.26920	3.71476	56
5	.21408	4.67121	.23240	4.30291	.25087	3.98607	.26951	3.71046	55
6	.21438	4.66458	.23271	4.29724	.25118	3.98117	.26982	3.70616	54
7	.21469	4.65797	.23301	4.29159	.25149	3.97627	.27013	3.70188	53
8	.21499	4.65138	.23332	4.28595	.25180	3.97139	.27044	3.69761	52
9	.21529	4.64480	.23363	4.28032	.25211	3.96651	.27076	3.69335	51
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	50
11	.21590	4.63171	.23424	4.26911	.25273	3.95680	.27138	3.68485	49
12	.21621	4.62518	.23455	4.26352	.25304	3.95196	.27169	3.68061	48
13	.21651	4.61868	.23485	4.25795	.25335	3.94713	.27201	3.67638	47
14	.21682	4.61219	.23516	4.25239	.25366	3.94232	.27232	3.67217	46
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	45
16	.21743	4.59927	.23578	4.24132	.25428	3.93271	.27294	3.66376	44
17	.21773	4.59283	.23608	4.23580	.25459	3.92793	.27326	3.65957	43
18	.21804	4.58641	.23639	4.23030	.25490	3.92316	.27357	3.65538	42
19	.21834	4.58001	.23670	4.22481	.25521	3.91839	.27388	3.65121	41
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	40
21	.21895	4.56726	.23731	4.21387	.25583	3.90890	.27451	3.64289	39
22	.21925	4.56091	.23762	4.20842	.25614	3.90417	.27482	3.63874	38
23	.21956	4.55458	.23793	4.20298	.25645	3.89945	.27513	3.63461	37
24	.21986	4.54826	.23823	4.19756	.25676	3.89474	.27545	3.63048	36
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	35
26	.22047	4.53568	.23885	4.18675	.25738	3.88536	.27607	3.62224	34
27	.22078	4.52941	.23916	4.18137	.25769	3.88068	.27638	3.61814	33
28	.22108	4.52316	.23946	4.17600	.25800	3.87601	.27670	3.61405	32
29	.22139	4.51693	.23977	4.17064	.25831	3.87136	.27701	3.60996	31
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3.60588	30
31	.22200	4.50451	.24039	4.15997	.25893	3.86208	.27764	3.60181	29
32	.22231	4.49832	.24069	4.15465	.25924	3.85745	.27795	3.59775	28
33	.22261	4.49215	.24100	4.14934	.25955	3.85284	.27826	3.59370	27
34	.22292	4.48600	.24131	4.14405	.25986	3.84824	.27858	3.58966	26
35	.22322	4.47986	.24162	4.13877	.26017	3.84364	.27889	3.58562	25
36	.22353	4.47374	.24193	4.13350	.26048	3.83906	.27920	3.58160	24
37	.22383	4.46764	.24223	4.12825	.26079	3.83449	.27952	3.57753	23
38	.22414	4.46155	.24254	4.12301	.26110	3.82992	.27983	3.57357	22
39	.22444	4.45548	.24285	4.11778	.26141	3.82537	.28015	3.56957	21
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	20
41	.22505	4.44338	.24347	4.10736	.26203	3.81630	.28077	3.56159	19
42	.22536	4.43735	.24377	4.10216	.26235	3.81177	.28109	3.55761	18
43	.22567	4.43134	.24408	4.09699	.26266	3.80726	.28140	3.55364	17
44	.22597	4.42534	.24439	4.09182	.26297	3.80276	.28172	3.54968	16
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	15
46	.22658	4.41340	.24501	4.08152	.26359	3.79378	.28234	3.54179	14
47	.22689	4.40745	.24532	4.07639	.26390	3.78931	.28266	3.53785	13
48	.22719	4.40152	.24562	4.07127	.26421	3.78485	.28297	3.53393	12
49	.22750	4.39560	.24593	4.06616	.26452	3.78040	.28329	3.53001	11
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	10
51	.22811	4.38381	.24655	4.05599	.26515	3.77152	.28391	3.52219	9
52	.22842	4.37793	.24686	4.05092	.26546	3.76709	.28423	3.51829	8
53	.22872	4.37207	.24717	4.04586	.26577	3.76268	.28454	3.51441	7
54	.22903	4.36623	.24747	4.04081	.26608	3.75828	.28486	3.51053	6
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	5
56	.22964	4.35459	.24809	4.03075	.26670	3.74950	.28549	3.50279	4
57	.22995	4.34879	.24840	4.02574	.26701	3.74512	.28580	3.49894	3
58	.23026	4.34300	.24871	4.02074	.26733	3.74075	.28612	3.49509	2
59	.23056	4.33723	.24902	4.01576	.26764	3.73640	.28643	3.49125	1
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	0
′	cotan 77°	tan 77°	cotan 76°	tan 76°	cotan 75°	tan 75°	cotan 74°	tan 74°	′

′	16° tan	16° cotan	17° tan	17° cotan	18° tan	18° cotan	19° tan	19° cotan	′
0	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35019	2.85555	42
19	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35117	2.84758	39
22	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94590	.35904	2.78523	15
46	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	73° cotan	73° tan	72° cotan	72° tan	71° cotan	71° tan	70° cotan	70° tan	′

′	20° tan	20° cotan	21° tan	21° cotan	22° tan	22° cotan	23° tan	23° cotan	′
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	39
22	.37124	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43239	2.31271	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	33
28	.37322	2.67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	27
34	.37521	2.66516	.29526	2.53001	.41558	2.40629	.43620	2.29254	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	18
43	.37820	2.64410	.39829	2.51076	.41865	2.33862	.43932	2.27626	17
44	.37853	2.64177	.39862	2.50864	.41899	2.38668	.43966	2.27447	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	5
56	.38253	2.61418	.40267	2.48340	.42310	2.36349	.44384	2.25309	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2.24780	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	0
′	69° cotan	69° tan	68° cotan	68° tan	67° cotan	67° tan	66° cotan	66° tan	′

′	24° tan	24° cotan	25° tan	25° cotan	26° tan	26° cotan	27° tan	27° cotan	′
0	.44523	2.24604	.46631	2.14451	.48773	2.05030	.50953	1.96261	60
1	.44558	2.24428	.46666	2.14288	.48809	2.04879	.50989	1.96120	59
2	.44593	2.24252	.46702	2.14125	.48845	2.04728	.51026	1.95979	58
3	.44627	2.24077	.46737	2.13963	.48881	2.04577	.51063	1.95838	57
4	.44662	2.23902	.46772	2.13801	.48917	2.04426	.51099	1.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	.44732	2.23553	.46843	2.13477	.48989	2.04125	.51173	1.95417	54
7	.44767	2.23378	.46879	2.13316	.49026	2.03975	.51209	1.95277	53
8	.44802	2.23204	.46914	2.13154	.49062	2.03825	.51246	1.95137	52
9	.44837	2.23030	.46950	2.12993	.49098	2.03675	.51283	1.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	.44907	2.22683	.47021	2.12671	.49170	2.03376	.51356	1.94718	49
12	.44942	2.22510	.47056	2.12511	.49206	2.03227	.51393	1.94579	48
13	.44977	2.22337	.47092	2.12350	.49242	2.03078	.51430	1.94440	47
14	.45012	2.22164	.47128	2.12190	.49278	2.02929	.51467	1.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	.45082	2.21819	.47199	2.11871	.49351	2.02631	.51540	1.94023	44
17	.45117	2.21647	.47234	2.11711	.49387	2.02483	.51577	1.93885	43
18	.45152	2.21475	.47270	2.11552	.49423	2.02335	.51614	1.93746	42
19	.45187	2.21304	.47305	2.11392	.49459	2.02187	.51651	1.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	.45257	2.20961	.47377	2.11075	.49532	2.01891	.51724	1.93332	39
22	.45292	2.20790	.47412	2.10916	.49568	2.01743	.51761	1.93195	38
23	.45327	2.20619	.47448	2.10758	.49604	2.01596	.51798	1.93057	37
24	.45362	2.20449	.47483	2.10600	.49640	2.01449	.51835	1.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	.45432	2.20108	.47555	2.10284	.49713	2.01155	.51909	1.92645	34
27	.45467	2.19938	.47590	2.10126	.49749	2.01008	.51946	1.92508	33
28	.45502	2.19769	.47626	2.09969	.49786	2.00862	.51983	1.92371	32
29	.45537	2.19599	.47662	2.09811	.49822	2.00715	.52020	1.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	.45608	2.19261	.47733	2.09498	.49894	2.00423	.52094	1.91962	29
32	.45643	2.19092	.47769	2.09341	.49931	2.00277	.52131	1.91826	28
33	.45678	2.18923	.47805	2.09184	.49967	2.00131	.52168	1.91690	27
34	.45713	2.18755	.47840	2.09028	.50004	1.99986	.52205	1.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	.45784	2.18419	.47912	2.08716	.50076	1.99695	.52279	1.91282	24
37	.45819	2.18251	.47948	2.08560	.50113	1.99550	.52316	1.91147	23
38	.45854	2.18084	.47984	2.08405	.50149	1.99406	.52353	1.91012	22
39	.45889	2.17916	.48019	2.08250	.50185	1.99261	.52390	1.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	.45960	2.17582	.48091	2.07939	.50258	1.98972	.52464	1.90607	19
42	.45995	2.17416	.48127	2.07785	.50295	1.98828	.52501	1.90472	18
43	.46030	2.17249	.48163	2.07630	.50331	1.98684	.52538	1.90337	17
44	.46065	2.17083	.48198	2.07476	.50368	1.98540	.52575	1.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	.46136	2.16751	.48270	2.07167	.50441	1.98253	.52650	1.89935	14
47	.46171	2.16585	.48306	2.07014	.50477	1.98110	.52687	1.89801	13
48	.46206	2.16420	.48342	2.06860	.50514	1.97966	.52724	1.89667	12
49	.46242	2.16255	.48378	2.06706	.50550	1.97823	.52761	1.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97680	.52798	1.89400	10
51	.46312	2.15925	.48450	2.06400	.50623	1.97538	.52836	1.89266	9
52	.46348	2.15760	.48486	2.06247	.50660	1.97395	.52873	1.89133	8
53	.46383	2.15596	.48521	2.06094	.50696	1.97253	.52910	1.89000	7
54	.46418	2.15432	.48557	2.05942	.50733	1.97111	.52947	1.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52984	1.88734	5
56	.46489	2.15104	.48629	2.05637	.50806	1.96827	.53022	1.88602	4
57	.46525	2.14940	.48665	2.05485	.50843	1.96685	.53059	1.88469	3
58	.46560	2.14777	.48701	2.05333	.50879	1.96544	.53096	1.88337	2
59	.46595	2.14614	.48737	2.05182	.50916	1.96402	.53134	1.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
′	cotan	tan	cotan	tan	cotan	tan	cotan	tan	′
	65°		64°		63°		62°		

′	28°		29°		30°		31°		′
	tan	cotan	tan	cotan	tan	cotan	tan	cotan	
0	.53171	1.88073	.55431	1.80405	.57735	1.73205	.60086	1.66428	60
1	.53208	1.87941	.55469	1.80281	.57774	1.73089	.60126	1.66318	59
2	.53246	1.87809	.55507	1.80158	.57813	1.72973	.60165	1.66209	58
3	.53283	1.87677	.55545	1.80034	.57851	1.72857	.60205	1.66099	57
4	.53320	1.87546	.55583	1.79911	.57890	1.72741	.60245	1.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	.53395	1.87283	.55659	1.79665	.57968	1.72509	.60324	1.65772	54
7	.53432	1.87152	.55697	1.79542	.58007	1.72393	.60364	1.65663	53
8	.53470	1.87021	.55736	1.79419	.58046	1.72278	.60403	1.65554	52
9	.53507	1.86891	.55774	1.79296	.58085	1.72163	.60443	1.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	.53582	1.86630	.55850	1.79051	.58162	1.71932	.60522	1.65228	49
12	.53620	1.86499	.55888	1.78929	.58201	1.71817	.60562	1.65120	48
13	.53657	1.86369	.55926	1.78807	.58240	1.71702	.60602	1.65011	47
14	.53694	1.86239	.55964	1.78685	.58279	1.71588	.60642	1.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	.53769	1.85979	.56041	1.78441	.58357	1.71358	.60721	1.64687	44
17	.53807	1.85850	.56079	1.78319	.58396	1.71244	.60761	1.64579	43
18	.53844	1.85720	.56117	1.78198	.58435	1.71129	.60801	1.64471	42
19	.53882	1.85591	.56156	1.78077	.58474	1.71015	.60841	1.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	.53957	1.85333	.56232	1.77834	.58552	1.70787	.60921	1.64148	39
22	.53995	1.85204	.56270	1.77713	.58591	1.70673	.60960	1.64041	38
23	.54032	1.85075	.56309	1.77592	.58631	1.70560	.61000	1.63934	37
24	.54070	1.84946	.56347	1.77471	.58670	1.70446	.61040	1.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	.54145	1.84689	.56424	1.77230	.58748	1.70219	.61120	1.63612	34
27	.54183	1.84561	.56462	1.77110	.58787	1.70106	.61160	1.63505	33
28	.54220	1.84433	.56500	1.76990	.58826	1.69992	.61200	1.63398	32
29	.54258	1.84305	.56539	1.76869	.58865	1.69879	.61240	1.63292	31
30	.54296	1.84177	.56577	1.76749	.58904	1.69766	.61280	1.63185	30
31	.54333	1.84049	.56616	1.76630	.58944	1.69653	.61320	1.63079	29
32	.54371	1.83922	.56654	1.76510	.58983	1.69541	.61360	1.62972	28
33	.54409	1.83794	.56693	1.76390	.59022	1.69428	.61400	1.62866	27
34	.54446	1.83667	.56731	1.76271	.59061	1.69316	.61440	1.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	.54522	1.83413	.56808	1.76032	.59140	1.69091	.61520	1.62548	24
37	.54560	1.83286	.56846	1.75913	.59179	1.68979	.61561	1.62442	23
38	.54597	1.83159	.56885	1.75794	.59218	1.68866	.61601	1.62336	22
39	.54635	1.83033	.56923	1.75675	.59258	1.68754	.61641	1.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	.54711	1.82780	.57000	1.75437	.59336	1.68531	.61721	1.62019	19
42	.54748	1.82654	.57039	1.75319	.59376	1.68419	.61761	1.61914	18
43	.54786	1.82528	.57078	1.75200	.59415	1.68308	.61801	1.61808	17
44	.54824	1.82402	.57116	1.75082	.59454	1.68196	.61842	1.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	.54900	1.82150	.57193	1.74846	.59533	1.67974	.61922	1.61493	14
47	.54938	1.82025	.57232	1.74728	.59573	1.67863	.61962	1.61388	13
48	.54975	1.81899	.57271	1.74610	.59612	1.67752	.62003	1.61283	12
49	.55013	1.81774	.57309	1.74492	.59651	1.67641	.62043	1.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	.55089	1.81524	.57386	1.74257	.59730	1.67419	.62124	1.60970	9
52	.55127	1.81399	.57425	1.74140	.59770	1.67309	.62164	1.60865	8
53	.55165	1.81274	.57464	1.74022	.59809	1.67198	.62204	1.60761	7
54	.55203	1.81150	.57503	1.73905	.59849	1.67088	.62245	1.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	.55279	1.80901	.57580	1.73671	.59928	1.66867	.62325	1.60449	4
57	.55317	1.80777	.57619	1.73555	.59967	1.66757	.62366	1.60345	3
58	.55355	1.80653	.57657	1.73438	.60007	1.66647	.62406	1.60241	2
59	.55393	1.80529	.57696	1.73321	.60046	1.66538	.62446	1.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
′	cotan	tan	cotan	tan	cotan	tan	cotan	tan	′
	61°		60°		59°		58°		

′	32° tan	32° cotan	33° tan	33° cotan	34° tan	34° cotan	35° tan	35° cotan	′
0	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	60
1	.62527	1.59930	.64982	1.53888	.67493	1.48163	.70064	1.42726	59
2	.62568	1.59826	.65023	1.53791	.67536	1.48070	.70107	1.42638	58
3	.62608	1.59723	.65065	1.53693	.67578	1.47977	.70151	1.42550	57
4	.62649	1.59620	.65106	1.53595	.67620	1.47885	.70194	1.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	.62730	1.59414	.65189	1.53400	.67705	1.47699	.70281	1.42286	54
7	.62770	1.59311	.65231	1.53302	.67748	1.47607	.70325	1.42198	53
8	.62811	1.59208	.65272	1.53205	.67790	1.47514	.70368	1.42110	52
9	.62852	1.59105	.65314	1.53107	.67832	1.47422	.70412	1.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	.62933	1.58900	.65397	1.52913	.67917	1.47238	.70499	1.41847	49
12	.62973	1.58797	.65438	1.52816	.67960	1.47146	.70542	1.41759	48
13	.63014	1.58695	.65480	1.52719	.68002	1.47053	.70586	1.41672	47
14	.63055	1.58593	.65521	1.52622	.68045	1.46962	.70629	1.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	.63136	1.58388	.65604	1.52429	.68130	1.46778	.70717	1.41409	44
17	.63177	1.58286	.65646	1.52332	.68173	1.46686	.70760	1.41322	43
18	.63217	1.58184	.65688	1.52235	.68215	1.46595	.70804	1.41235	42
19	.63258	1.58083	.65729	1.52139	.68258	1.46503	.70848	1.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	.63340	1.57879	.65813	1.51946	.68343	1.46320	.70935	1.40974	39
22	.63380	1.57778	.65854	1.51850	.68386	1.46229	.70979	1.40887	38
23	.63421	1.57676	.65896	1.51754	.68429	1.46137	.71023	1.40800	37
24	.63462	1.57575	.65938	1.51658	.68471	1.46046	.71066	1.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	.63544	1.57372	.66021	1.51466	.68557	1.45864	.71154	1.40540	34
27	.63584	1.57271	.66063	1.51370	.68600	1.45773	.71198	1.40454	33
28	.63625	1.57170	.66105	1.51275	.68642	1.45682	.71242	1.40367	32
29	.63666	1.57069	.66147	1.51179	.68685	1.45592	.71285	1.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	.63748	1.56868	.66230	1.50988	.68771	1.45410	.71373	1.40109	29
32	.63789	1.56767	.66272	1.50893	.68814	1.45320	.71417	1.40022	28
33	.63830	1.56667	.66314	1.50799	.68857	1.45229	.71461	1.39936	27
34	.63871	1.56566	.66356	1.50702	.68900	1.45138	.71505	1.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	.63953	1.56366	.66440	1.50512	.68985	1.44958	.71593	1.39679	24
37	.63994	1.56265	.66482	1.50417	.69028	1.44868	.71637	1.39593	23
38	.64035	1.56165	.66524	1.50322	.69071	1.44778	.71681	1.39507	22
39	.64076	1.56065	.66566	1.50228	.69114	1.44688	.71725	1.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	.64158	1.55866	.66650	1.50038	.69200	1.44508	.71813	1.39250	19
42	.64199	1.55766	.66692	1.49944	.69243	1.44418	.71857	1.39165	18
43	.64240	1.55666	.66734	1.49849	.69286	1.44329	.71901	1.39079	17
44	.64281	1.55567	.66776	1.49755	.69329	1.44239	.71946	1.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	.64363	1.55368	.66860	1.49566	.69416	1.44060	.72034	1.38824	14
47	.64404	1.55269	.66902	1.49472	.69459	1.43970	.72078	1.38738	13
48	.64446	1.55170	.66944	1.49378	.69502	1.43881	.72122	1.38653	12
49	.64487	1.55071	.66986	1.49284	.69545	1.43792	.72166	1.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	.64569	1.54873	.67071	1.49097	.69631	1.43614	.72255	1.38399	9
52	.64610	1.54774	.67113	1.49003	.69675	1.43525	.72299	1.38314	8
53	.64652	1.54675	.67155	1.48909	.69718	1.43436	.72344	1.38229	7
54	.64693	1.54576	.67197	1.48816	.69761	1.43347	.72388	1.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	.64775	1.54379	.67282	1.48629	.69847	1.43169	.72477	1.37976	4
57	.64817	1.54281	.67324	1.48536	.69891	1.43080	.72521	1.37891	3
58	.64858	1.54183	.67366	1.48442	.69934	1.42992	.72565	1.37807	2
59	.64899	1.54085	.67409	1.48349	.69966	1.42903	.72610	1.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72654	1.37638	0
′	57° cotan	57° tan	56° cotan	56° tan	55° cotan	55° tan	54° cotan	54° tan	′

′	36° tan	36° cotan	37° tan	37° cotan	38° tan	38° cotan	39° tan	39° cotan	′
0	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	60
1	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	59
2	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	58
3	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	57
4	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	56
5	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	54
7	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	53
8	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	52
9	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	51
10	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	49
12	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	48
13	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	47
14	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	46
15	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	44
17	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	43
18	.73457	1.36133	.76180	1.31269	.78975	1.26622	.81849	1.22176	42
19	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	41
20	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	39
22	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	38
23	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	37
24	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	36
25	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	34
27	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	33
28	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	32
29	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	31
30	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	29
32	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	28
33	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	27
34	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	26
35	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	24
37	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	23
38	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	22
39	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	21
40	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	19
42	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	18
43	.74583	1.34070	.77335	1.29307	.80163	1.24746	.83071	1.20379	17
44	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	16
45	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	14
47	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	13
48	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	12
49	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	11
50	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	9
52	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	8
53	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	7
54	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	6
55	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	4
57	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	3
58	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	2
59	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	1
60	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	0
′	53° cotan	53° tan	52° cotan	52° tan	51° cotan	51° tan	50° cotan	50° tan	′

′	40° tan	40° cotan	41° tan	41° cotan	42° tan	42° cotan	43° tan	43° cotan	′
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	33
28	.85307	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	0
′	cotan 49°	tan 49°	cotan 48°	tan 48°	cotan 47°	tan 47°	cotan 46°	tan 46°	′

′	44° tan	cotan	′	′	44° tan	cotan	′	′	44° tan	cotan	′
0	.96569	1.03553	60	21	.97756	1.02295	39	41	.98901	1.01112	19
1	.96625	1.03493	59	22	.97813	1.02236	38	42	.98958	1.01053	18
2	.96681	1.03433	58	23	.97870	1.02176	37	43	.99016	1.00994	17
3	.96738	1.03372	57	24	.97927	1.02117	36	44	.99073	1.00935	16
4	.96794	1.03312	56	25	.97984	1.02057	35	45	.99131	1.00876	15
5	.96850	1.03252	55	26	.98041	1.01998	34	46	.99189	1.00818	14
6	.96907	1.03192	54	27	.98098	1.01939	33	47	.99247	1.00759	13
7	.96963	1.03132	53	28	.98155	1.01879	32	48	.99304	1.00701	12
8	.97020	1.03072	52	29	.98213	1.01820	31	49	.99362	1.00642	11
9	.97076	1.03012	51	30	.98270	1.01761	30	50	.99420	1.00583	10
10	.97133	1.02952	50	31	.98327	1.01702	29	51	.99478	1.00525	9
11	.97189	1.02892	49	32	.98384	1.01642	28	52	.99536	1.00467	8
12	.97246	1.02832	48	33	.98441	1.01583	27	53	.99594	1.00408	7
13	.97302	1.02772	47	34	.98499	1.01524	26	54	.99652	1.00350	6
14	.97359	1.02713	46	35	.98556	1.01465	25	55	.99710	1.00291	5
15	.97416	1.02653	45	36	.98613	1.01406	24	56	.99768	1.00233	4
16	.97472	1.02593	44	37	.98671	1.01347	23	57	.99826	1.00175	3
17	.97529	1.02533	43	38	.98728	1.01288	22	58	.99884	1.00116	2
18	.97586	1.02474	42	39	.98786	1.01229	21	59	.99942	1.00058	1
19	.97643	1.02414	41	40	.98843	1.01170	20	60	1	1	0
20	.97700	1.02355	40								
′	cotan 45°	tan	′	′	cotan 45°	tan	′	′	cotan 45°	tan	′

NATURAL SINES AND COSINES

′	0° sine	cosine	′	′	0° sine	cosine	′	′	0° sine	cosine	′
0	.00000	1	60	21	.00611	.99998	39	41	.01193	.99993	19
1	.00029	1	59	22	.00640	.99998	38	42	.01222	.99993	18
2	.00058	1	58	23	.00669	.99998	37	43	.01251	.99992	17
3	.00087	1	57	24	.00698	.99998	36	44	.01280	.99992	16
4	.00116	1	56	25	.00727	.99997	35	45	.01309	.99991	15
5	.00145	1	55	26	.00756	.99997	34	46	.01338	.99991	14
6	.00175	1	54	27	.00785	.99997	33	47	.01367	.99991	13
7	.00204	1	53	28	.00814	.99997	32	48	.01396	.99990	12
8	.00233	1	52	29	.00844	.99996	31	49	.01425	.99990	11
9	.00262	1	51	30	.00873	.99996	30	50	.01454	.99989	10
10	.00291	1	50	31	.00902	.99996	29	51	.01483	.99989	9
11	.00320	.99999	49	32	.00931	.99996	28	52	.01513	.99989	8
12	.00349	.99999	48	33	.00960	.99995	27	53	.01542	.99988	7
13	.00378	.99999	47	34	.00989	.99995	26	54	.01571	.99988	6
14	.00407	.99999	46	35	.01018	.99995	25	55	.01600	.99987	5
15	.00436	.99999	45	36	.01047	.99995	24	56	.01629	.99987	4
16	.00465	.99999	44	37	.01076	.99994	23	57	.01658	.99986	3
17	.00495	.99999	43	38	.01105	.99994	22	58	.01687	.99986	2
18	.00524	.99999	42	39	.01134	.99994	21	59	.01716	.99985	1
19	.00553	.99998	41	40	.01164	.99993	20	60	.01745	.99985	0
20	.00582	.99998	40								
′	cosine 89°	sine	′	′	cosine 89°	sine	′	′	cosine 89°	sine	′

′	1° sine	1° cosine	2° sine	2° cosine	3° sine	3° cosine	4° sine	4° cosine	′
0	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	88° cosine	88° sine	87° cosine	87° sine	86° cosine	86° sine	85° cosine	85° sine	′

′	5°		6°		7°		8°		′
	sine	cosine	sine	cosine	sine	cosine	sine	cosine	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	0
′	cosine	sine	cosine	sine	cosine	sine	cosine	sine	′
	84°		83°		82°		81°		

′	9° sine	9° cosine	10° sine	10° cosine	11° sine	11° cosine	12° sine	12° cosine	′
0	.15643	.98769	.17365	.98481	.19081	.98163	.20791	.97815	60
1	.15672	.98764	.17393	.98476	.19109	.98157	.20820	.97809	59
2	.15701	.98760	.17422	.98471	.19138	.98152	.20848	.97803	58
3	.15730	.98755	.17451	.98466	.19167	.98146	.20877	.97797	57
4	.15758	.98751	.17479	.98461	.19195	.98140	.20905	.97791	56
5	.15787	.98746	.17508	.98455	.19224	.98135	.20933	.97784	55
6	.15816	.98741	.17537	.98450	.19252	.98129	.20962	.97778	54
7	.15845	.98737	.17565	.98445	.19281	.98124	.20990	.97772	53
8	.15873	.98732	.17594	.98440	.19309	.98118	.21019	.97766	52
9	.15902	.98728	.17623	.98435	.19338	.98112	.21047	.97760	51
10	.15931	.98723	.17651	.98430	.19366	.98107	.21076	.97754	50
11	.15959	.98718	.17680	.98425	.19395	.98101	.21104	.97748	49
12	.15988	.98714	.17708	.98420	.19423	.98096	.21132	.97742	48
13	.16017	.98709	.17737	.98414	.19452	.98090	.21161	.97735	47
14	.16046	.98704	.17766	.98409	.19481	.98084	.21189	.97729	46
15	.16074	.98700	.17794	.98404	.19509	.98079	.21218	.97723	45
16	.16103	.98695	.17823	.98399	.19538	.98073	.21246	.97717	44
17	.16132	.98690	.17852	.98394	.19566	.98067	.21275	.97711	43
18	.16160	.98686	.17880	.98389	.19595	.98061	.21303	.97705	42
19	.16189	.98681	.17909	.98383	.19623	.98056	.21331	.97698	41
20	.16218	.98676	.17937	.98378	.19652	.98050	.21360	.97692	40
21	.16246	.98671	.17966	.98373	.19680	.98044	.21388	.97686	39
22	.16275	.98667	.17995	.98368	.19709	.98039	.21417	.97680	38
23	.16304	.98662	.18023	.98362	.19737	.98033	.21445	.97673	37
24	.16333	.98657	.18052	.98357	.19766	.98027	.21474	.97667	36
25	.16361	.98652	.18081	.98352	.19794	.98021	.21502	.97661	35
26	.16390	.98648	.18109	.98347	.19823	.98016	.21530	.97655	34
27	.16419	.98643	.18138	.98341	.19851	.98010	.21559	.97648	33
28	.16447	.98638	.18166	.98336	.19880	.98004	.21587	.97642	32
29	.16476	.98633	.18195	.98331	.19908	.97997	.21616	.97636	31
30	.16505	.98629	.18224	.98325	.19937	.97992	.21644	.97630	30
31	.16533	.98624	.18252	.98320	.19965	.97987	.21672	.97623	29
32	.16562	.98619	.18281	.98315	.19994	.97981	.21701	.97617	28
33	.16591	.98614	.18309	.98310	.20022	.97975	.21729	.97611	27
34	.16620	.98609	.18338	.98304	.20051	.97969	.21758	.97604	26
35	.16648	.98604	.18367	.98299	.20079	.97963	.21786	.97598	25
36	.16677	.98600	.18395	.98294	.20108	.97958	.21814	.97592	24
37	.16706	.98595	.18424	.98288	.20136	.97952	.21843	.97585	23
38	.16734	.98590	.18452	.98283	.20165	.97946	.21871	.97579	22
39	.16763	.98585	.18481	.98277	.20193	.97940	.21899	.97573	21
40	.16792	.98580	.18509	.98272	.20222	.97934	.21928	.97566	20
41	.16820	.98575	.18538	.98267	.20250	.97928	.21956	.97560	19
42	.16849	.98570	.18567	.98261	.20279	.97922	.21985	.97553	18
43	.16878	.98565	.18595	.98256	.20307	.97916	.22013	.97547	17
44	.16906	.98561	.18624	.98250	.20336	.97910	.22041	.97541	16
45	.16935	.98556	.18652	.98245	.20364	.97905	.22070	.97534	15
46	.16964	.98551	.18681	.98240	.20393	.97899	.22098	.97528	14
47	.16992	.98546	.18710	.98234	.20421	.97893	.22126	.97521	13
48	.17021	.98541	.18738	.98229	.20450	.97887	.22155	.97515	12
49	.17050	.98536	.18767	.98223	.20478	.97881	.22183	.97508	11
50	.17078	.98531	.18795	.98218	.20507	.97875	.22212	.97502	10
51	.17107	.98526	.18824	.98212	.20535	.97869	.22240	.97496	9
52	.17136	.98521	.18852	.98207	.20563	.97863	.22268	.97489	8
53	.17164	.98516	.18881	.98201	.20592	.97857	.22297	.97483	7
54	.17193	.98511	.18910	.98196	.20620	.97851	.22325	.97476	6
55	.17222	.98506	.18938	.98190	.20649	.97845	.22353	.97470	5
56	.17250	.98501	.18967	.98185	.20677	.97839	.22382	.97463	4
57	.17279	.98496	.18995	.98179	.20706	.97833	.22410	.97457	3
58	.17308	.98491	.19024	.98174	.20734	.97827	.22438	.97450	2
59	.17336	.98486	.19052	.98168	.20763	.97821	.22467	.97444	1
60	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	0
′	80° cosine	80° sine	79° cosine	79° sine	78° cosine	78° sine	77° cosine	77° sine	′

′	13° sine	13° cosine	14° sine	14° cosine	15° sine	15° cosine	16° sine	16° cosine	′
0	.22495	.97437	.24192	.97030	.25882	.96593	.27564	.96126	60
1	.22523	.97430	.24220	.97023	.25910	.96585	.27592	.96118	59
2	.22552	.97424	.24249	.97015	.25938	.96578	.27620	.86110	58
3	.22580	.97417	.24277	.97008	.25966	.96570	.27648	.96102	57
4	.22608	.97411	.24305	.97001	.25994	.96562	.27676	.96094	56
5	.22637	.97404	.24333	.96994	.26022	.96555	.27704	.96086	55
6	.22665	.97398	.24362	.96987	.26050	.96547	.27731	.96078	54
7	.22693	.97391	.24390	.96980	.26079	.96540	.27759	.96070	53
8	.22722	.97384	.24418	.96973	.26107	.96532	.27787	.96062	52
9	.22750	.97378	.24446	.96966	.26135	.96524	.27815	.96054	51
10	.22778	.97371	.24474	.96959	.26163	.96517	.27843	.96046	50
11	.22807	.97365	.24503	.96952	.26191	.96509	.27871	.96037	49
12	.22835	.97358	.24531	.96945	.26219	.96502	.27899	.96029	48
13	.22863	.97351	.24559	.96937	.26247	.96494	.27927	.96021	47
14	.22892	.97345	.24587	.96930	.26275	.96486	.27955	.96013	46
15	.22920	.97338	.24615	.96923	.26303	.96479	.27983	.96005	45
16	.22948	.97331	.24644	.96916	.26331	.96471	.28011	.95997	44
17	.22977	.97325	.24672	.96909	.26359	.96463	.28039	.95989	43
18	.23005	.97318	.24700	.96902	.26387	.96456	.28067	.95981	42
19	.23033	.97311	.24728	.96894	.26415	.96448	.28095	.95972	41
20	.23062	.97304	.24756	.96887	.26443	.96440	.28123	.95964	40
21	.23090	.97298	.24784	.96880	.26471	.96433	.28150	.95956	39
22	.23118	.97291	.24813	.96873	.26500	.96425	.28178	.95948	38
23	.23146	.97284	.24841	.96866	.26528	.96417	.28206	.95940	37
24	.23175	.97278	.24869	.96858	.26556	.96410	.28234	.95931	36
25	.23203	.97271	.24897	.96851	.26584	.96402	.28262	.95923	35
26	.23231	.97264	.24925	.96844	.26612	.96394	.28290	.95915	34
27	.23260	.97257	.24954	.96837	.26640	.96386	.28318	.95907	33
28	.23288	.97251	.24982	.96829	.26668	.96379	.28346	.95898	32
29	.23316	.97244	.25010	.96822	.26696	.96371	.28374	.95890	31
30	.23345	.97237	.25038	.96815	.26724	.96363	.28402	.95882	30
31	.23373	.97230	.25066	.96807	.26752	.96355	.28429	.95874	29
32	.23401	.97223	.25094	.96800	.26780	.96347	.28457	.95865	28
33	.23429	.97217	.25122	.96793	.26808	.96340	.28485	.95857	27
34	.23458	.07210	.25151	.96786	.26836	.96332	.28513	.95849	26
35	.23486	.97203	.25179	.96778	.26864	.96324	.28541	.95841	25
36	.23514	.97196	.25207	.96771	.26892	.96316	.28569	.95832	24
37	.23542	.97189	.25235	.96764	.26920	.96308	.28597	.95824	23
38	.23571	.97182	.25263	.96756	.26948	.96301	.28625	.95816	22
39	.23599	.97176	.25291	.96749	.26976	.96293	.28652	.95807	21
40	.23627	.97169	.25320	.96742	.27004	.96285	.28680	.95799	20
41	.23656	.97162	.25348	.96734	.27032	.96277	.28708	.95791	19
42	.23684	.97155	.25376	.96727	.27060	.96269	.28736	.95782	18
43	.23712	.97148	.25404	.96719	.27088	.96261	.28764	.95774	17
44	.23740	.97141	.25432	.96712	.27116	.96253	.28792	.95766	16
45	.23769	.97134	.25460	.96705	.27144	.96246	.28820	.95757	15
46	.23797	.97127	.25488	.96697	.27172	.96238	.28847	.95749	14
47	.23825	.97120	.25516	.96690	.27200	.96230	.28875	.95740	13
48	.23853	.97113	.25545	.96682	.27228	.96222	.28903	.95732	12
49	.23882	.97106	.25573	.96675	.27256	.96214	.28931	.95724	11
50	.23910	.97100	.25601	.96667	.27284	.96206	.28959	.95715	10
51	.23938	.97093	.25629	.96660	.27312	.96198	.28987	.95707	9
52	.23966	.97086	.25657	.96653	.27340	.96190	.29015	.95698	8
53	.23995	.97079	.25685	.96645	.27368	.96182	.29042	.95690	7
54	.24023	.97072	.25713	.96638	.27396	.96174	.29070	.95681	6
55	.24051	.97065	.25741	.96630	.27424	.96166	.29098	.95673	5
56	.24079	.97058	.25769	.96623	.27452	.96158	.29126	.95664	4
57	.24108	.97051	.25798	.96615	.27480	.96150	.29154	.95656	3
58	.24136	.97044	.25826	.96608	.27508	.96142	.29182	.95647	2
59	.24164	.97037	.25854	.96600	.27536	.96134	.29209	.95639	1
60	.24192	.97030	.25882	.96593	.27564	.96126	.29237	.95630	0
′	cosine	sine	cosine	sine	cosine	sine	cosine	sine	′
	76°		75°		74°		73°		

′	17° sine	17° cosine	18° sine	18° cosine	19° sine	19° cosine	20° sine	20° cosine	′
0	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	60
1	.29265	.95622	.30929	.95097	.32584	.94542	.34229	.93959	59
2	.29293	.95613	.30957	.95088	.32612	.94533	.34257	.93949	58
3	.29321	.95605	.30985	.95079	.32639	.94523	.34284	.93939	57
4	.29348	.95596	.31012	.95070	.32667	.94514	.34311	.93929	56
5	.29376	.95588	.31040	.95061	.32694	.94504	.34339	.93919	55
6	.29404	.95579	.31068	.95052	.32722	.94495	.34366	.93909	54
7	.29432	.95571	.31095	.95043	.32749	.94485	.34393	.93899	53
8	.29460	.95562	.31123	.95033	.32777	.94476	.34421	.93889	52
9	.29487	.95554	.31151	.95024	.32804	.94466	.34448	.93879	51
10	.29515	.95545	.31178	.95015	.32832	.94457	.34475	.93869	50
11	.29543	.95536	.31206	.95006	.32859	.94447	.34503	.93859	49
12	.29571	.95528	.31233	.94997	.32887	.94438	.34530	.93849	48
13	.29599	.95519	.31261	.94988	.32914	.94428	.34557	.93839	47
14	.29626	.95511	.31289	.94979	.32942	.94418	.34584	.93829	46
15	.29654	.95502	.31316	.94970	.32969	.94409	.34612	.93819	45
16	.29682	.95493	.31344	.94961	.32997	.94399	.34639	.93809	44
17	.29710	.95485	.31372	.94952	.33024	.94390	.34666	.93799	43
18	.29737	.95476	.31399	.94943	.33051	.94380	.34694	.93789	42
19	.29765	.95467	.31427	.94933	.33079	.94370	.34721	.93779	41
20	.29793	.95459	.31454	.94924	.33106	.94361	.34748	.93769	40
21	.29821	.95450	.31482	.94915	.33134	.94351	.34775	.93759	39
22	.29849	.95441	.31510	.94906	.33161	.94342	.34803	.93748	38
23	.29876	.95433	.31537	.94897	.33189	.94332	.34830	.93738	37
24	.29904	.95424	.31565	.94888	.33216	.94322	.34857	.93728	36
25	.29932	.95415	.31593	.94878	.33244	.94313	.34884	.93718	35
26	.29960	.95407	.31620	.94869	.33271	.94303	.34912	.93708	34
27	.29987	.95398	.31648	.94860	.33298	.94293	.34939	.93698	33
28	.30015	.95389	.31675	.94851	.33326	.94284	.34966	.93688	32
29	.30043	.05380	.31703	.94842	.33353	.94274	.34993	.93677	31
30	.30071	.95372	.31730	.94832	.33381	.94264	.35021	.93667	30
31	.30098	.95363	.31758	.94823	.33408	.94254	.35048	.93657	29
32	.30126	.95354	.31786	.94814	.33436	.94245	.35075	.93647	28
33	.30154	.95345	.31813	.94805	.33463	.94235	.35102	.93637	27
34	.30182	.95337	.31841	.94795	.33490	.94225	.35130	.93626	26
35	.30209	.95328	.31868	.94786	.33518	.94215	.35157	.93616	25
36	.30237	.95319	.31896	.94777	.33545	.94206	.35184	.93606	24
37	.30265	.95310	.31923	.94768	.33573	.94196	.35211	.93596	23
38	.30292	.95301	.31951	.94758	.33600	.94186	.35239	.93585	22
39	.30320	.95293	.31979	.94749	.33627	.94176	.35266	.93575	21
40	.30348	.95284	.32006	.94740	.33655	.94167	.35293	.93565	20
41	.30376	.95275	.32034	.94730	.33682	.94157	.35320	.93555	19
42	.30403	.95266	.32061	.94721	.33710	.94147	.35347	.93544	18
43	.30431	.95257	.32089	.94712	.33737	.94137	.35375	.93534	17
44	.30459	.95248	.32116	.94702	.33764	.94127	.35402	.93524	16
45	.30486	.95240	.32144	.94693	.33792	.94118	.35429	.93514	15
46	.30514	.95231	.32171	.94684	.33819	.94108	.35456	.93503	14
47	.30542	.95222	.32199	.94674	.33846	.94098	.35484	.93493	13
48	.30570	.95213	.32227	.94665	.33874	.94088	.35511	.93483	12
49	.30597	.95204	.32254	.94656	.33901	.94078	.35538	.93472	11
50	.30625	.95195	.32282	.94646	.33929	.94068	.35565	.93462	10
51	.30653	.95186	.32309	.94637	.33956	.94058	.35592	.93452	9
52	.30680	.95177	.32337	.94627	.33983	.94049	.35619	.93441	8
53	.30708	.95168	.32364	.94618	.34011	.94039	.35647	.93431	7
54	.30736	.95159	.32392	.94609	.34038	.94029	.35674	.93420	6
55	.30763	.95150	.32419	.94599	.34065	.94019	.35701	.93410	5
56	.30791	.95142	.32447	.94590	.34093	.94009	.35728	.93400	4
57	.30819	.95133	.32474	.94580	.34120	.93999	.35755	.93389	3
58	.30846	.95124	.32502	.94571	.34147	.93989	.35782	.93379	2
59	.30874	.95115	.32529	.94561	.34175	.93979	.35810	.93368	1
60	.30902	.95106	.32557	.94552	.34202	.93969	.35837	.93358	0
′	cosine 72°	sine 72°	cosine 71°	sine 71°	cosine 70°	sine 70°	cosine 69°	sine 69°	′

′	21°		22°		23°		24°		
	sine	cosine	sine	cosine	sine	cosine	sine	cosine	
0	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	cosine	sine	cosine	sine	cosine	sine	cosine	sine	′
	68°		67°		66°		65°		

′	25°		26°		27°		28°		′
	sine	cosine	sine	cosine	sine	cosine	sine	cosine	
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	0
′	cosine	sine	cosine	sine	cosine	sine	cosine	sine	′
	64°		63°		62°		61°		

′	29° sine	29° cosine	30° sine	30° cosine	31° sine	31° cosine	32° sine	32° cosine	′
0	.48481	.87462	.50000	.86603	.51504	.85717	.52992	.84805	60
1	.48506	.87448	.50025	.86588	.51529	.85702	.53017	.84789	59
2	.48532	.87434	.50050	.86573	.51554	.85687	.53041	.84774	58
3	.48557	.87420	.50076	.86559	.51579	.85672	.53066	.84759	57
4	.48583	.87406	.50101	.86544	.51604	.85657	.53091	.84743	56
5	.48608	.87391	.50126	.86530	.51628	.85642	.53115	.84728	55
6	.48634	.87377	.50151	.86515	.51653	.85627	.53140	.84712	54
7	.48659	.87363	.50176	.86501	.51678	.85612	.53164	.84697	53
8	.48684	.87349	.50201	.86486	.51703	.85597	.53189	.84681	52
9	.48710	.87335	.50227	.86471	.51728	.85582	.53214	.84666	51
10	.48735	.87321	.50252	.86457	.51753	.85567	.53238	.84650	50
11	.48761	.87306	.50277	.86442	.51778	.85551	.53263	.84635	49
12	.48786	.87292	.50302	.86427	.51803	.85536	.53288	.84619	48
13	.48811	.87278	.50327	.86413	.51828	.85521	.53312	.84604	47
14	.48837	.87264	.50352	.86398	.51852	.85506	.53337	.84588	46
15	.48862	.87250	.50377	.86384	.51877	.85491	.53361	.84573	45
16	.48888	.87235	.50403	.86369	.51902	.85476	.53386	.84557	44
17	.48913	.87221	.50428	.86354	.51927	.85461	.53411	.84542	43
18	.48938	.87207	.50453	.86340	.51952	.85446	.53435	.84526	42
19	.48964	.87193	.50478	.86325	.51977	.85431	.53460	.84511	41
20	.48989	.87178	.50503	.86310	.52002	.85416	.53484	.84495	40
21	.49014	.87164	.50528	.86295	.52026	.85401	.53509	.84480	39
22	.49040	.87150	.50553	.86281	.52051	.85385	.53534	.84464	38
23	.49065	.87136	.50578	.86266	.52076	.85370	.53558	.84448	37
24	.49090	.87121	.50603	.86251	.52101	.85355	.53583	.84433	36
25	.49116	.87107	.50628	.86237	.52126	.85340	.53607	.84417	35
26	.49141	.87093	.50654	.86222	.52151	.85325	.53632	.84402	34
27	.49166	.87079	.50679	.86207	.52175	.85310	.53656	.84386	33
28	.49192	.87064	.50704	.86192	.52200	.85294	.53681	.84370	32
29	.49217	.87050	.50729	.86178	.52225	.85279	.53705	.84355	31
30	.49242	.87036	.50754	.86163	.52250	.85264	.53730	.84339	30
31	.49268	.87021	.50779	.86148	.52275	.85249	.53754	.84324	29
32	.49293	.87007	.50804	.86133	.52299	.85234	.53779	.84308	28
33	.49318	.86993	.50829	.86119	.52324	.85218	.53804	.84292	27
34	.49344	.86978	.50854	.86104	.52349	.85203	.53828	.84277	26
35	.49369	.86964	.50879	.86089	.52374	.85188	.53853	.84261	25
36	.49394	.86949	.50904	.86074	.52399	.85173	.53877	.84245	24
37	.49419	.86935	.50929	.86059	.52423	.85157	.53902	.84230	23
38	.49445	.86921	.50954	.86045	.52448	.85142	.53926	.84214	22
39	.49470	.86906	.50979	.86030	.52473	.85127	.53951	.84198	21
40	.49495	.86892	.51004	.86015	.52498	.85112	.53975	.84182	20
41	.49521	.86878	.51029	.86000	.52522	.85096	.54000	.84167	19
42	.49546	.86863	.51054	.85985	.52547	.85081	.54024	.84151	18
43	.49571	.86849	.51079	.85970	.52572	.85066	.54049	.84135	17
44	.49596	.86834	.51104	.85956	.52597	.85051	.54073	.84120	16
45	.49622	.86820	.51129	.85941	.52621	.85035	.54097	.84104	15
46	.49647	.86805	.51154	.85926	.52646	.85020	.54122	.84088	14
47	.49672	.86791	.51179	.85911	.52671	.85005	.54146	.84072	13
48	.49697	.86777	.51204	.85896	.52696	.84989	.54171	.84057	12
49	.49723	.86762	.51229	.85881	.52720	.84974	.54195	.84041	11
50	.49748	.86748	.51254	.85866	.52745	.84959	.54220	.84025	10
51	.49773	.86733	.51279	.85851	.52770	.84943	.54244	.84009	9
52	.49798	.86719	.51304	.85836	.52794	.84928	.54269	.83994	8
53	.49824	.86704	.51329	.85821	.52819	.84913	.54293	.83978	7
54	.49849	.86690	.51354	.85806	.52844	.84897	.54317	.83962	6
55	.49874	.86675	.51379	.85792	.52869	.84882	.54342	.83946	5
56	.49899	.86661	.51404	.85777	.52893	.84866	.54366	.83930	4
57	.49924	.86646	.51429	.85762	.52918	.84851	.54391	.83915	3
58	.49950	.86632	.51454	.85747	.52943	.84836	.54415	.83899	2
59	.49975	.86617	.51479	.85732	.52967	.84820	.54440	.83883	1
60	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	0
′	cosine 60°	sine 60°	cosine 59°	sine 59°	cosine 58°	sine 58°	cosine 57°	sine 57°	′

′	33° sine	33° cosine	34° sine	34° cosine	35° sine	35° cosine	36° sine	36° cosine	′
0	.54464	.83867	.55919	.82904	.57358	.81915	.58779	.80902	60
1	.54488	.83851	.55943	.82887	.57381	.81899	.58802	.80885	59
2	.54513	.83835	.55968	.82871	.57405	.81882	.58826	.80867	58
3	.54537	.83819	.55992	.82855	.57429	.81865	.58849	.80850	57
4	.54561	.83804	.56016	.82839	.57453	.81848	.58873	.80833	56
5	.54586	.83788	.56040	.82822	.57477	.81832	.58896	.80816	55
6	.54610	.83772	.56064	.82806	.57501	.81815	.58920	.80799	54
7	.54635	.83756	.56088	.82790	.57524	.81798	.58943	.80782	53
8	.54659	.83740	.56112	.82773	.57548	.81782	.58967	.80765	52
9	.54683	.83724	.56136	.82757	.57572	.81765	.58990	.80748	51
10	.54708	.83708	.56160	.82741	.57596	.81748	.59014	.80730	50
11	.54732	.83692	.56184	.82724	.57619	.81731	.59037	.80713	49
12	.54756	.83676	.56208	.82708	.57643	.81714	.59061	.80696	48
13	.54781	.83660	.56232	.82692	.57667	.81698	.59084	.80679	47
14	.54805	.83645	.56256	.82675	.57691	.81681	.59108	.80662	46
15	.54829	.83629	.56280	.82659	.57715	.81664	.59131	.80644	45
16	.54854	.83613	.56305	.82643	.57738	.81647	.59154	.80627	44
17	.54878	.83597	.56329	.82626	.57762	.81631	.59178	.80610	43
18	.54902	.83581	.56353	.82610	.57786	.81614	.59201	.80593	42
19	.54927	.83565	.56377	.82593	.57810	.81597	.59225	.80576	41
20	.54951	.83549	.56401	.82577	.57833	.81580	.59248	.80558	40
21	.54975	.83533	.56425	.82561	.57857	.81563	.59272	.80541	39
22	.54999	.83517	.56449	.82544	.57881	.81546	.59295	.80524	38
23	.55024	.83501	.56473	.82528	.57904	.81530	.59318	.80507	37
24	.55048	.83485	.56497	.82511	.57928	.81513	.59342	.80489	36
25	.55072	.83469	.56521	.82495	.57952	.81496	.59365	.80472	35
26	.55097	.83453	.56545	.82478	.57976	.81479	.59389	.80455	34
27	.55121	.83437	.56569	.82462	.57999	.81462	.59412	.80438	33
28	.55145	.83421	.56593	.82446	.58023	.81445	.59436	.80420	32
29	.55169	.83405	.56617	.82429	.58047	.81428	.59459	.80403	31
30	.55194	.83389	.56641	.82413	.58070	.81412	.59482	.80386	30
31	.55218	.83373	.56665	.82396	.58094	.81395	.59506	.80368	29
32	.55242	.83356	.56689	.82380	.58118	.81378	.59529	.80351	28
33	.55266	.83340	.56713	.82363	.58141	.81361	.59552	.80334	27
34	.55291	.83324	.56736	.82347	.58165	.81344	.59576	.80316	26
35	.55315	.83308	.56760	.82330	.58189	.81327	.59599	.80299	25
36	.55339	.83292	.56784	.82314	.58212	.81310	.59622	.80282	24
37	.55363	.83276	.56808	.82297	.58236	.81293	.59646	.80264	23
38	.55388	.83260	.56832	.82281	.58260	.81276	.59669	.80247	22
39	.55412	.83244	.56856	.82264	.58283	.81259	.59693	.80230	21
40	.55436	.83228	.56880	.82248	.58307	.81242	.59716	.80212	20
41	.55460	.83212	.56904	.82231	.58330	.81225	.59739	.80195	19
42	.55484	.83195	.56928	.82214	.58354	.81208	.59763	.80178	18
43	.55509	.83179	.56952	.82198	.58378	.81191	.59786	.80160	17
44	.55533	.83163	.56976	.82181	.58401	.81174	.59809	.80143	16
45	.55557	.83147	.57000	.82165	.58425	.81157	.59832	.80125	15
46	.55581	.83131	.57024	.82148	.58449	.81140	.59856	.80108	14
47	.55605	.83115	.57047	.82132	.58472	.81123	.59879	.80091	13
48	.55630	.83098	.57071	.82115	.58496	.81106	.59902	.80073	12
49	.55654	.83082	.57095	.82098	.58519	.81089	.59926	.80056	11
50	.55678	.83066	.57119	.82082	.58543	.81072	.59949	.80038	10
51	.55702	.83050	.57143	.82065	.58567	.81055	.59972	.80021	9
52	.55726	.83034	.57167	.82048	.58590	.81038	.59995	.80003	8
53	.55750	.83017	.57191	.82032	.58614	.81021	.60019	.79986	7
54	.55775	.83001	.57215	.82015	.58637	.81004	.60042	.79968	6
55	.55799	.82985	.57238	.81999	.58661	.80987	.60065	.79951	5
56	.55823	.82969	.57262	.81982	.58684	.80970	.60089	.79934	4
57	.55847	.82953	.57286	.81965	.58708	.80953	.60112	.79916	3
58	.55871	.82936	.57310	.81949	.58731	.80936	.60135	.79899	2
59	.55895	.82920	.57334	.81932	.58755	.80919	.60158	.79881	1
60	.55919	.82904	.57358	.81915	.58779	.80902	.60182	.79864	0
′	56° cosine	56° sine	55° cosine	55° sine	54° cosine	54° sine	53° cosine	53° sine	′

′	37° sine	37° cosine	38° sine	38° cosine	39° sine	39° cosine	40° sine	40° cosine	′
0	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	60
1	.60205	.79846	.61589	.78783	.62955	.77696	.64301	.76586	59
2	.60228	.79829	.61612	.78765	.62977	.77678	.64323	.76567	58
3	.60251	.79811	.61635	.78747	.63000	.77660	.64346	.76548	57
4	.60274	.79793	.61658	.78729	.63022	.77641	.64368	.76530	56
5	.60298	.79776	.61681	.78711	.63045	.77623	.64390	.76511	55
6	.60321	.79758	.61704	.78694	.63068	.77605	.64412	.76492	54
7	.60344	.79741	.61726	.78676	.63090	.77586	.64435	.76473	53
8	.60367	.79723	.61749	.78658	.63113	.77568	.64457	.76455	52
9	.60390	.79706	.61772	.78640	.63135	.77550	.64479	.76436	51
10	.60414	.79688	.61795	.78622	.63158	.77531	.64501	.76417	50
11	.60437	.79671	.61818	.78604	.63180	.77513	.64524	.76398	49
12	.60460	.79653	.61841	.78586	.63203	.77494	.64546	.76380	48
13	.60483	.79635	.61864	.78568	.63225	.77476	.64568	.76361	47
14	.60506	.79618	.61887	.78550	.63248	.77458	.64590	.76342	46
15	.60529	.79600	.61909	.78532	.63271	.77439	.64612	.76323	45
16	.60553	.79583	.61932	.78514	.63293	.77421	.64635	.76304	44
17	.60576	.79565	.61955	.78496	.63316	.77402	.64657	.76286	43
18	.60599	.79547	.61978	.78478	.63338	.77384	.64679	.76267	42
19	.60622	.79530	.62001	.78460	.63361	.77366	.64701	.76248	41
20	.60645	.79512	.62024	.78442	.63383	.77347	.64723	.76229	40
21	.60668	.79494	.62046	.78424	.63406	.77329	.64746	.76210	39
22	.60691	.79477	.62069	.78405	.63428	.77310	.64768	.76192	38
23	.60714	.79459	.62092	.78387	.63451	.77292	.64790	.76173	37
24	.60738	.79441	.62115	.78369	.63473	.77273	.64812	.76154	36
25	.60761	.79424	.62138	.78351	.63496	.77255	.64834	.76135	35
26	.60784	.79406	.62160	.78333	.63518	.77236	.64856	.76116	34
27	.60807	.79388	.62183	.78315	.63540	.77218	.64878	.76097	33
28	.60830	.79371	.62206	.78297	.63563	.77199	.64901	.76078	32
29	.60853	.79353	.62229	.78279	.63585	.77181	.64923	.76059	31
30	.60876	.79335	.62251	.78261	.63608	.77162	.64945	.76041	30
31	.60899	.79318	.62274	.78243	.63630	.77144	.64967	.76022	29
32	.60922	.79300	.62297	.78225	.63653	.77125	.64989	.76003	28
33	.60945	.79282	.62320	.78206	.63675	.77107	.65011	.75984	27
34	.60968	.79264	.62342	.78188	.63698	.77088	.65033	.75965	26
35	.60991	.79247	.62365	.78170	.63720	.77070	.65055	.75946	25
36	.61015	.79229	.62388	.78152	.63742	.77051	.65077	.75927	24
37	.61038	.79211	.62411	.78134	.63765	.77033	.65100	.75908	23
38	.61061	.79193	.62433	.78116	.63787	.77014	.65122	.75889	22
39	.61084	.79176	.62456	.78098	.63810	.76996	.65144	.75870	21
40	.61107	.79158	.62479	.78079	.63832	.76977	.65166	.75851	20
41	.61130	.79140	.62502	.78061	.63854	.76959	.65188	.75832	19
42	.61153	.79122	.62524	.78043	.63877	.76940	.65210	.75813	18
43	.61176	.79105	.62547	.78025	.63899	.76921	.65232	.75794	17
44	.61199	.79087	.62570	.78007	.63922	.76903	.65254	.75775	16
45	.61222	.79069	.62592	.77988	.63944	.76884	.65276	.75756	15
46	.61245	.79051	.62615	.77970	.63966	.76866	.65298	.75738	14
47	.61268	.79033	.62638	.77952	.63989	.76847	.65320	.75719	13
48	.61291	.79016	.62660	.77934	.64011	.76828	.65342	.75700	12
49	.61314	.78998	.62683	.77916	.64033	.76810	.65364	.75680	11
50	.61337	.78980	.62706	.77897	.64056	.76791	.65386	.75661	10
51	.61360	.78962	.62728	.77879	.64078	.76772	.65408	.75642	9
52	.61383	.78944	.62751	.77861	.64100	.76754	.65430	.75623	8
53	.61406	.78926	.62774	.77843	.64123	.76735	.65452	.75604	7
54	.61429	.78908	.62796	.77824	.64145	.76717	.65474	.75585	6
55	.61451	.78891	.62819	.77806	.64167	.76698	.65496	.75566	5
56	.61474	.78873	.62842	.77788	.64190	.76679	.65518	.75547	4
57	.61497	.78855	.62864	.77769	.64212	.76661	.65540	.75528	3
58	.61520	.78837	.62887	.77751	.64234	.76642	.65562	.75509	2
59	.61543	.78819	.62909	.77733	.64256	.76623	.65584	.75490	1
60	.61566	.78801	.62932	.77715	.64279	.76604	.65606	.75471	0
′	cosine 52°	sine 52°	cosine 51°	sine 51°	cosine 50°	sine 50°	cosine 49°	sine 49°	′

′	41°		42°		43°		44°		′
	sine	cosine	sine	cosine	sine	cosine	sine	cosine	
0	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	cosine	sine	cosine	sine	cosine	sine	cosine	sine	′
	48°		47°		46°		45°		

′	0° sec	0° cosec	1° sec	1° cosec	2° sec	2° cosec	3° sec	3° cosec	′
0	1	Infinite.	1.0001	57.299	1.0006	28.654	1.0014	19.107	60
1	1	3437.70	1.0001	56.359	1.0006	28.417	1.0014	19.002	59
2	1	1718.90	1.0002	55.450	1.0006	28.184	1.0014	18.897	58
3	1	1145.90	1.0002	54.570	1.0006	27.955	1.0014	18.794	57
4	1	859.44	1.0002	53.718	1.0006	27.730	1.0014	18.692	56
5	1	687.55	1.0002	52.891	1.0007	27.508	1.0014	18.591	55
6	1	572.96	1.0002	52.090	1.0007	27.290	1.0015	18.491	54
7	1	491.11	1.0002	51.313	1.0007	27.075	1.0015	18.393	53
8	1	429.72	1.0002	50.558	1.0007	26.864	1.0015	18.295	52
9	1	381.97	1.0002	49.826	1.0007	26.655	1.0015	18.198	51
10	1	343.77	1.0002	49.114	1.0007	26.450	1.0015	18.103	50
11	1	312.52	1.0002	48.422	1.0007	26.249	1.0015	18.008	49
12	1	236.48	1.0002	47.750	1.0007	26.050	1.0016	17.914	48
13	1	264.44	1.0002	47.096	1.0007	25.854	1.0016	17.821	47
14	1	245.55	1.0002	46.460	1.0008	25.661	1.0016	17.730	46
15	1	229.18	1.0002	45.840	1.0008	25.471	1.0016	17.639	45
16	1	214.86	1.0002	45.237	1.0008	25.284	1.0016	17.549	44
17	1	202.22	1.0002	44.650	1.0008	25.100	1.0016	17.460	43
18	1	190.99	1.0002	44.077	1.0008	24.918	1.0017	17.372	42
19	1	180.73	1.0003	43.520	1.0008	24.739	1.0017	17.285	41
20	1	171.89	1.0003	42.976	1.0008	24.562	1.0017	17.198	40
21	1	163.70	1.0003	42.445	1.0008	24.388	1.0017	17.113	39
22	1	156.26	1.0003	41.928	1.0008	24.216	1.0017	17.028	38
23	1	149.47	1.0003	41.423	1.0009	24.047	1.0017	16.944	37
24	1	143.24	1.0003	40.930	1.0009	23.880	1.0018	16.861	36
25	1	137.51	1.0003	40.448	1.0009	23.716	1.0018	16.779	35
26	1	132.22	1.0003	39.978	1.0009	23.553	1.0018	16.698	34
27	1	127.32	1.0003	39.518	1.0009	23.393	1.0018	16.617	33
28	1	122.78	1.0003	39.069	1.0009	23.235	1.0018	16.538	32
29	1	118.54	1.0003	38.631	1.0009	23.079	1.0018	16.459	31
30	1	114.59	1.0003	38.201	1.0009	22.925	1.0019	16.380	30
31	1	110.90	1.0003	37.782	1.0010	22.774	1.0019	16.303	29
32	1	107.43	1.0003	37.371	1.0010	22.624	1.0019	16.226	28
33	1	104.17	1.0004	36.969	1.0010	22.476	1.0019	16.150	27
34	1	101.11	1.0004	36.576	1.0010	22.330	1.0019	16.075	26
35	1	98.223	1.0004	36.191	1.0010	22.186	1.0019	16.000	25
36	1	95.495	1.0004	35.814	1.0010	22.044	1.0020	15.926	24
37	1	92.914	1.0004	35.445	1.0010	21.904	1.0020	15.853	23
38	1.0001	90.469	1.0004	35.084	1.0010	21.765	1.0020	15.780	22
39	1.0001	88.149	1.0004	34.729	1.0011	21.629	1.0020	15.708	21
40	1.0001	85.946	1.0004	34.382	1.0011	21.494	1.0020	15.637	20
41	1.0001	83.849	1.0004	34.042	1.0011	21.360	1.0021	15.566	19
42	1.0001	81.853	1.0004	33.708	1.0011	21.228	1.0021	15.496	18
43	1.0001	79.950	1.0004	33.381	1.0011	21.098	1.0021	15.427	17
44	1.0001	78.133	1.0004	33.060	1.0011	20.970	1.0021	15.358	16
45	1.0001	76.396	1.0005	32.745	1.0011	20.843	1.0021	15.290	15
46	1.0001	74.736	1.0005	32.437	1.0012	20.717	1.0022	15.222	14
47	1.0001	73.146	1.0005	32.134	1.0012	20.593	1.0022	15.155	13
48	1.0001	71.622	1.0005	31.836	1.0012	20.471	1.0022	15.089	12
49	1.0001	70.160	1.0005	31.544	1.0012	20.350	1.0022	15.023	11
50	1.0001	68.757	1.0005	31.257	1.0012	20.230	1.0022	14.958	10
51	1.0001	67.409	1.0005	30.976	1.0012	20.112	1.0023	14.893	9
52	1.0001	66.113	1.0005	30.699	1.0012	19.995	1.0023	14.829	8
53	1.0001	64.866	1.0005	30.428	1.0013	19.880	1.0023	14.765	7
54	1.0001	63.664	1.0005	30.161	1.0013	19.766	1.0023	14.702	6
55	1.0001	62.507	1.0005	29.899	1.0013	19.653	1.0023	14.640	5
56	1.0001	61.391	1.0006	29.641	1.0013	19.541	1.0024	14.578	4
57	1.0001	60.314	1.0006	29.388	1.0013	19.431	1.0024	14.517	3
58	1.0001	59.274	1.0006	29.139	1.0013	19.322	1.0024	14.456	2
59	1.0001	58.270	1.0006	28.894	1.0013	19.214	1.0024	14.395	1
60	1.0001	57.299	1.0006	28.654	1.0014	19.107	1.0024	14.335	0
′	89° cosec	89° sec	88° cosec	88° sec	87° cosec	87° sec	86° cosec	86° sec	′

′	4° sec	4° cosec	5° sec	5° cosec	6° sec	6° cosec	7° sec	7° cosec	′
0	1.0024	14.335	1.0038	11.474	1.0055	9.5668	1.0075	8.2055	60
1	1.0025	14.276	1.0038	11.436	1.0055	9.5404	1.0075	8.1861	59
2	1.0025	14.217	1.0039	11.398	1.0056	9.5141	1.0076	8.1668	58
3	1.0025	14.159	1.0039	11.360	1.0056	9.4880	1.0076	8.1476	57
4	1.0025	14.101	1.0039	11.323	1.0056	9.4620	1.0076	8.1285	56
5	1.0025	14.043	1.0039	11.286	1.0057	9.4362	1.0077	8.1094	55
6	1.0026	13.986	1.0040	11.249	1.0057	9.4105	1.0077	8.0905	54
7	1.0026	13.930	1.0040	11.213	1.0057	9.3850	1.0078	8.0717	53
8	1.0026	13.874	1.0040	11.176	1.0057	9.3596	1.0078	8.0529	52
9	1.0026	13.818	1.0040	11.140	1.0058	9.3343	1.0078	8.0342	51
10	1.0026	13.763	1.0041	11.104	1.0058	9.3092	1.0079	8.0156	50
11	1.0027	13.708	1.0041	11.069	1.0058	9.2842	1.0079	7.9971	49
12	1.0027	13.654	1.0041	11.033	1.0059	9.2598	1.0079	7.9787	48
13	1.0027	13.600	1.0041	10.988	1.0059	9.2346	1.0080	7.9604	47
14	1.0027	13.547	1.0042	10.963	1.0059	9.2100	1.0080	7.9421	46
15	1.0027	13.494	1.0042	10.929	1.0060	9.1855	1.0080	7.9240	45
16	1.0028	13.441	1.0042	10.894	1.0060	9.1612	1.0081	7.9059	44
17	1.0028	13.389	1.0043	10.860	1.0060	9.1370	1.0081	7.8879	43
18	1.0028	13.337	1.0043	10.826	1.0061	9.1129	1.0082	7.8700	42
19	1.0028	13.286	1.0043	10.792	1.0061	9.0890	1.0082	7.8522	41
20	1.0029	13.235	1.0043	10.758	1.0061	9.0651	1.0082	7.8344	40
21	1.0029	13.184	1.0044	10.725	1.0062	9.0414	1.0083	7.8168	39
22	1.0029	13.134	1.0044	10.692	1.0062	9.0179	1.0083	7.7992	38
23	1.0029	13.084	1.0044	10.659	1.0062	8.9944	1.0084	7.7817	37
24	1.0029	13.034	1.0044	10.626	1.0063	8.9711	1.0084	7.7642	36
25	1.0030	12.985	1.0045	10.593	1.0063	8.9479	1.0084	7.7469	35
26	1.0030	12.937	1.0045	10.561	1.0063	8.9248	1.0085	7.7296	34
27	1.0030	12.888	1.0045	10.529	1.0064	8.9018	1.0085	7.7124	33
28	1.0030	12.840	1.0046	10.497	1.0064	8.8790	1.0085	7.6953	32
29	1.0031	12.793	1.0046	10.465	1.0064	8.8563	1.0086	7.6783	31
30	1.0031	12.745	1.0046	10.433	1.0065	8.8337	1.0086	7.6613	30
31	1.0031	12.698	1.0046	10.402	1.0065	8.8112	1.0087	7.6444	29
32	1.0031	12.652	1.0047	10.371	1.0065	8.7888	1.0087	7.6276	28
33	1.0032	12.606	1.0047	10.340	1.0066	8.7665	1.0087	7.6108	27
34	1.0032	12.560	1.0047	10.309	1.0066	8.7444	1.0088	7.5942	26
35	1.0032	12.514	1.0048	10.278	1.0066	8.7223	1.0088	7.5776	25
36	1.0032	12.469	1.0048	10.248	1.0067	8.7004	1.0089	7.5611	24
37	1.0032	12.424	1.0048	10.217	1.0067	8.6786	1.0089	7.5446	23
38	1.0033	12.379	1.0048	10.187	1.0067	8.6569	1.0089	7.5282	22
39	1.0033	12.335	1.0049	10.157	1.0068	8.6353	1.0090	7.5119	21
40	1.0033	12.291	1.0049	10.127	1.0068	8.6138	1.0090	7.4957	20
41	1.0033	12.248	1.0049	10.098	1.0068	8.5924	1.0090	7.4795	19
42	1.0034	12.204	1.0050	10.068	1.0069	8.5711	1.0091	7.4634	18
43	1.0034	12.161	1.0050	10.039	1.0069	8.5499	1.0091	7.4474	17
44	1.0034	12.118	1.0050	10.010	1.0069	8.5289	1.0092	7.4315	16
45	1.0034	12.076	1.0050	9.9812	1.0070	8.5079	1.0092	7.4156	15
46	1.0035	12.034	1.0051	9.9525	1.0070	8.4871	1.0092	7.3998	14
47	1.0035	11.992	1.0051	9.9239	1.0070	8.4663	1.0093	7.3840	13
48	1.0035	11.950	1.0051	9.8955	1.0071	8.4457	1.0093	7.3683	12
49	1.0035	11.909	1.0052	9.8672	1.0071	8.4251	1.0094	7.3527	11
50	1.0036	11.868	1.0052	9.8391	1.0071	8.4046	1.0094	7.3372	10
51	1.0036	11.828	1.0052	9.8112	1.0072	8.3843	1.0094	7.3217	9
52	1.0036	11.787	1.0053	9.7834	1.0072	8.3640	1.0095	7.3063	8
53	1.0036	11.747	1.0053	9.7558	1.0073	8.3439	1.0095	7.2909	7
54	1.0037	11.707	1.0053	9.7283	1.0073	8.3238	1.0096	7.2757	6
55	1.0037	11.668	1.0053	9.7010	1.0073	8.3039	1.0096	7.2604	5
56	1.0037	11.628	1.0054	9.6739	1.0074	8.2840	1.0097	7.2453	4
57	1.0037	11.589	1.0054	9.6469	1.0074	8.2642	1.0097	7.2302	3
58	1.0038	11.550	1.0054	9.6200	1.0074	8.2446	1.0097	7.2152	2
59	1.0038	11.512	1.0055	9.5933	1.0075	8.2250	1.0098	7.2002	1
60	1.0038	11.474	1.0055	9.5668	1.0075	8.2055	1.0098	7.1853	0
′	cosec	sec	cosec	sec	cosec	sec	cosec	sec	′
	85°		84°		83°		82°		

′	8° sec	8° cosec	9° sec	9° cosec	10° sec	10° cosec	11° sec	11° cosec	′
0	1.0098	7.1853	1.0125	6.3924	1.0154	5.7588	1.0187	5.2408	60
1	1.0099	7.1704	1.0125	6.3807	1.0155	5.7493	1.0188	5.2330	59
2	1.0099	7.1557	1.0125	6.3690	1.0155	5.7398	1.0188	5.2252	58
3	1.0099	7.1409	1.0126	6.3574	1.0156	5.7304	1.0189	5.2174	57
4	1.0100	7.1263	1.0126	6.3458	1.0156	5.7210	1.0189	5.2097	56
5	1.0100	7.1117	1.0127	6.3343	1.0157	5.7117	1.0190	5.2019	55
6	1.0101	7.0972	1.0127	6.3228	1.0157	5.7023	1.0191	5.1942	54
7	1.0101	7.0827	1.0128	6.3113	1.0158	5.6930	1.0191	5.1865	53
8	1.0102	7.0683	1.0128	6.2999	1.0158	5.6838	1.0192	5.1788	52
9	1.0102	7.0539	1.0129	6.2885	1.0159	5.6745	1.0192	5.1712	51
10	1.0102	7.0396	1.0129	6.2772	1.0159	5.6653	1.0193	5.1636	50
11	1.0103	7.0254	1.0130	6.2659	1.0160	5.6561	1.0193	5.1560	49
12	1.0103	7.0112	1.0130	6.2546	1.0160	5.6470	1.0194	5.1484	48
13	1.0104	6.9971	1.0131	6.2434	1.0161	5.6379	1.0195	5.1409	47
14	1.0104	6.9830	1.0131	6.2322	1.0162	5.6288	1.0195	5.1333	46
15	1.0104	6.9690	1.0132	6.2211	1.0162	5.6197	1.0196	5.1258	45
16	1.0105	6.9550	1.0132	6.2100	1.0163	5.6107	1.0196	5.1183	44
17	1.0105	6.9411	1.0133	6.1990	1.0163	5.6017	1.0197	5.1109	43
18	1.0106	6.9273	1.0133	6.1880	1.0164	5.5928	1.0198	5.1034	42
19	1.0106	6.9135	1.0134	6.1770	1.0164	5.5838	1.0198	5.0960	41
20	1.0107	6.8998	1.0134	6.1661	1.0165	5.5749	1.0199	5.0886	40
21	1.0107	6.8861	1.0135	6.1552	1.0165	5.5660	1.0199	5.0812	39
22	1.0107	6.8725	1.0135	6.1443	1.0166	5.5572	1.0200	5.0739	38
23	1.0108	6.8589	1.0136	6.1335	1.0166	5.5484	1.0201	5.0666	37
24	1.0108	6.8454	1.0136	6.1227	1.0167	5.5396	1.0201	5.0593	36
25	1.0109	6.8320	1.0136	6.1120	1.0167	5.5308	1.0202	5.0520	35
26	1.0109	6.8185	1.0137	6.1013	1.0168	5.5221	1.0202	5.0447	34
27	1.0110	6.8052	1.0137	6.0906	1.0169	5.5134	1.0203	5.0375	33
28	1.0110	6.7919	1.0138	6.0800	1.0169	5.5047	1.0204	5.0302	32
29	1.0111	6.7787	1.0138	6.0694	1.0170	5.4960	1.0204	5.0230	31
30	1.0111	6.7655	1.0139	6.0588	1.0170	5.4874	1.0205	5.0158	30
31	1.0111	6.7523	1.0139	6.0483	1.0171	5.4788	1.0205	5.0087	29
32	1.0112	6.7392	1.0140	6.0379	1.0171	5.4702	1.0206	5.0015	28
33	1.0112	6.7262	1.0140	6.0274	1.0172	5.4617	1.0207	4.9944	27
34	1.0113	6.7132	1.0141	6.0170	1.0172	5.4532	1.0207	4.9873	26
35	1.0113	6.7003	1.0141	6.0066	1.0173	5.4447	1.0208	4.9802	25
36	1.0114	6.6874	1.0142	5.9963	1.0174	5.4362	1.0208	4.9732	24
37	1.0114	6.6745	1.0142	5.9860	1.0174	5.4278	1.0209	4.9661	23
38	1.0115	6.6617	1.0143	5.9758	1.0175	5.4194	1.0210	4.9591	22
39	1.0115	6.6490	1.0143	5.9655	1.0175	5.4110	1.0210	4.9521	21
40	1.0015	6.6363	1.0144	5.9554	1.0176	5.4026	1.0211	4.9452	20
41	1.0116	6.6237	1.0144	5.9452	1.0176	5.3943	1.0211	4.9382	19
42	1.0116	6.6111	1.0145	5.9351	1.0177	5.3860	1.0212	4.9313	18
43	1.0117	6.5985	1.0145	5.9250	1.0177	5.3777	1.0213	4.9243	17
44	1.0117	6.5860	1.0146	5.9150	1.0178	5.3695	1.0213	4.9175	16
45	1.0118	6.5736	1.0146	5.9049	1.0179	5.3612	1.0214	4.9106	15
46	1.0118	6.5612	1.0147	5.8950	1.0179	5.3530	1.0215	4.9037	14
47	1.0119	6.5488	1.0147	5.8850	1.0180	5.3449	1.0215	4.8969	13
48	1.0119	6.5365	1.0148	5.8751	1.0180	5.3367	1.0216	4.8901	12
49	1.0119	6.5243	1.0148	5.8652	1.0181	5.3286	1.0216	4.8833	11
50	1.0120	6.5121	1.0149	5.8554	1.0181	5.3205	1.0217	4.8765	10
51	1.0120	6.4999	1.0150	5.8456	1.0182	5.3124	1.0218	4.8697	9
52	1.0121	6.4878	1.0150	5.8358	1.0182	5.3044	1.0218	4.8630	8
53	1.0121	6.4757	1.0151	5.8261	1.0183	5.2963	1.0219	4.8563	7
54	1.0122	6.4637	1.0151	5.8163	1.0184	5.2883	1.0220	4.8496	6
55	1.0122	6.4517	1.0152	5.8067	1.0184	5.2803	1.0220	4.8429	5
56	1.0123	6.4398	1.0152	5.7970	1.0185	5.2724	1.0221	4.8362	4
57	1.0123	6.4279	1.0153	5.7874	1.0185	5.2645	1.0221	4.8296	3
58	1.0124	6.4160	1.0153	5.7778	1.0186	5.2566	1.0222	4.8229	2
59	1.0124	6.4042	1.0154	5.7683	1.0186	5.2487	1.0223	4.8163	1
60	1.0125	6.3924	1.0154	5.7588	1.0187	5.2408	1.0223	4.8097	0
′	cosec 81°	sec 81°	cosec 80°	sec 80°	cosec 79°	sec 79°	cosec 78°	sec 78°	′

′	12° sec	12° cosec	13° sec	13° cosec	14° sec	14° cosec	15° sec	15° cosec	′
0	1.0223	4.8097	1.0263	4.4454	1.0306	4.1336	1.0353	3.8637	60
1	1.0224	4.8032	1.0264	4.4398	1.0307	4.1287	1.0353	3.8595	59
2	1.0225	4.7966	1.0264	4.4342	1.0308	4.1239	1.0354	3.8553	58
3	1.0225	4.7901	1.0265	4.4287	1.0308	4.1191	1.0355	3.8512	57
4	1.0226	4.7835	1.0266	4.4231	1.0309	4.1144	1.0356	3.8470	56
5	1.0226	4.7770	1.0266	4.4176	1.0310	4.1096	1.0357	3.8428	55
6	1.0227	4.7706	1.0267	4.4121	1.0311	4.1048	1.0358	3.8387	54
7	1.0228	4.7641	1.0268	4.4065	1.0311	4.1001	1.0358	3.8346	53
8	1.0228	4.7576	1.0268	4.4011	1.0312	4.0953	1.0359	3.8304	52
9	1.0229	4.7512	1.0269	4.3956	1.0313	4.0906	1.0360	3.8263	51
10	1.0230	4.7448	1.0270	4.3901	1.0314	4.0859	1.0361	3.8222	50
11	1.0230	4.7384	1.0271	4.3847	1.0314	4.0812	1.0362	3.8181	49
12	1.0231	4.7320	1.0271	4.3792	1.0315	4.0765	1.0362	3.8140	48
13	1.0232	4.7257	1.0272	4.3738	1.0316	4.0718	1.0363	3.8100	47
14	1.0232	4.7193	1.0273	4.3684	1.0317	4.0672	1.0364	3.8059	46
15	1.0233	4.7130	1.0273	4.3630	1.0317	4.0625	1.0365	3.8018	45
16	1.0234	4.7067	1.0274	4.3576	1.0318	4.0579	1.0366	3.7978	44
17	1.0234	4.7004	1.0275	4.3522	1.0319	4.0532	1.0367	3.7937	43
18	1.0235	4.6942	1.0276	4.3469	1.0320	4.0486	1.0367	3.7897	42
19	1.0235	4.6879	1.0276	4.3415	1.0320	4.0440	1.0368	3.7857	41
20	1.0236	4.6817	1.0277	4.3362	1.0321	4.0394	1.0369	3.7816	40
21	1.0237	4.6754	1.0278	4.3309	1.0322	4.0348	1.0370	3.7776	39
22	1.0237	4.6692	1.0278	4.3256	1.0323	4.0302	1.0371	3.7736	38
23	1.0238	4.6631	1.0279	4.3203	1.0323	4.0256	1.0371	3.7697	37
24	1.0239	4.6569	1.0280	4.3150	1.0324	4.0211	1.0372	3.7657	36
25	1.0239	4.6507	1.0280	4.3098	1.0325	4.0165	1.0373	3.7617	35
26	1.0240	4.6446	1.0281	4.3045	1.0326	4.0120	1.0374	3.7577	34
27	1.0241	4.6385	1.0282	4.2993	1.0327	4.0074	1.0375	3.7538	33
28	1.0241	4.6324	1.0283	4.2941	1.0327	4.0029	1.0376	3.7498	32
29	1.0242	4.6263	1.0283	4.2388	1.0328	3.9984	1.0376	3.7459	31
30	1.0243	4.6202	1.0284	4.2836	1.0329	3.9939	1.0377	3.7420	30
31	1.0243	4.6142	1.0285	4.2785	1.0330	3.9894	1.0378	3.7380	29
32	1.0244	4.6081	1.0285	4.2733	1.0330	3.9850	1.0379	3.7341	28
33	1.0245	4.6021	1.0286	4.2681	1.0331	3.9805	1.0380	3.7302	27
34	1.0245	4.5961	1.0287	4.2630	1.0332	3.9760	1.0381	3.7263	26
35	1.0246	4.5901	1.0288	4.2579	1.0333	3.9716	1.0382	3.7224	25
36	1.0247	4.5841	1.0288	4.2527	1.0334	3.9672	1.0382	3.7186	24
37	1.0247	4.5782	1.0289	4.2476	1.0334	3.9027	1.0383	3.7147	23
38	1.0248	4.5722	1.0290	4.2425	1.0335	3.9533	1.0384	3.7108	22
39	1.0249	4.5663	1.0291	4.2375	1.0336	3.9539	1.0385	3.7070	21
40	1.0249	4.5604	1.0291	4.2324	1.0337	3.9495	1.0386	3.7031	20
41	1.0250	4.5545	1.0292	4.2273	1.0338	3.9451	1.0387	3.6993	19
42	1.0251	4.5486	1.0293	4.2223	1.0338	3.9408	1.0387	3.6955	18
43	1.0251	4.5428	1.0293	4.2173	1.0339	3.9364	1.0388	3.6917	17
44	1.0252	4.5369	1.0294	4.2122	1.0340	3.9320	1.0389	3.6878	16
45	1.0253	4.5311	1.0295	4.2072	1.0341	3.9277	1.0390	3.6840	15
46	1.0253	4.5253	1.0296	4.2022	1.0341	3.9234	1.0391	3.6802	14
47	1.0254	4.5195	1.0296	4.1972	1.0342	3.9190	1.0392	3.6765	13
48	1.0255	4.5137	1.0297	4.1923	1.0343	3.9147	1.0393	3.6727	12
49	1.0255	4.5079	1.0298	4.1873	1.0344	3.9104	1.0393	3.6689	11
50	1.0256	4.5021	1.0299	4.1824	1.0345	3.9061	1.0394	3.6651	10
51	1.0257	4.4964	1.0299	4.1774	1.0345	3.9018	1.0395	3.6614	9
52	1.0257	4.4907	1.0300	4.1725	1.0346	3.8976	1.0396	3.6576	8
53	1.0258	4.4850	1.0301	4.1676	1.0347	3.8933	1.0397	3.6539	7
54	1.0259	4.4793	1.0302	4.1627	1.0348	3.8890	1.0398	3.6502	6
55	1.0260	4.4736	1.0302	4.1578	1.0348	3.8848	1.0399	3.6464	5
56	1.0260	4.4679	1.0303	4.1529	1.0349	3.8805	1.0399	3.6427	4
57	1.0261	4.4623	1.0304	4.1481	1.0350	3.8763	1.0400	3.6390	3
58	1.0262	4.4566	1.0305	4.1432	1.0351	3.8721	1.0401	3.6353	2
59	1.0262	4.4510	1.0305	4.1384	1.0352	3.8679	1.0402	3.6316	1
60	1.0263	4.4454	1.0306	4.1336	1.0353	3.8637	1.0403	3.6279	0
′	cosec 77°	sec 77°	cosec 76°	sec 76°	cosec 75°	sec 75°	cosec 74°	sec 74°	′

′	16° sec	16° cosec	17° sec	17° cosec	18° sec	18° cosec	19° sec	19° cosec	′
0	1.0403	3.6279	1.0457	3.4203	1.0515	3.2361	1.0576	3.0715	60
1	1.0404	3.6243	1.0458	3.4170	1.0516	3.2332	1.0577	3.0690	59
2	1.0405	3.6206	1.0459	3.4138	1.0517	3.2303	1.0578	3.0664	58
3	1.0406	3.6169	1.0460	3.4106	1.0518	3.2274	1.0579	3.0638	57
4	1.0406	3.6133	1.0461	3.4073	1.0519	3.2245	1.0580	3.0612	56
5	1.0407	3.6096	1.0461	3.4041	1.0520	3.2216	1.0581	3.0586	55
6	1.0408	3.6060	1.0462	3.4009	1.0521	3.2188	1.0582	3.0561	54
7	1.0409	3.6024	1.0463	3.3977	1.0522	3.2159	1.0584	3.0535	53
8	1.0410	3.5987	1.0464	3.3945	1.0523	3.2131	1.0585	3.0509	52
9	1.0411	3.5951	1.0465	3.3913	1.0524	3.2102	1.0586	3.0484	51
10	1.0412	3.5915	1.0466	3.3881	1.0525	3.2074	1.0587	3.0458	50
11	1.0413	3.5879	1.0467	3.3849	1.0526	3.2045	1.0588	3.0433	49
12	1.0413	3.5843	1.0468	3.3817	1.0527	3.2017	1.0589	3.0407	48
13	1.0414	3.5807	1.0469	3.3785	1.0528	3.1989	1.0590	3.0382	47
14	1.0415	3.5772	1.0470	3.3754	1.0529	3.1960	1.0591	3.0357	46
15	1.0416	3.5736	1.0471	3.3722	1.0530	3.1932	1.0592	3.0331	45
16	1.0417	3.5700	1.0472	3.3690	1.0531	3.1904	1.0593	3.0306	44
17	1.0418	3.5665	1.0473	3.3659	1.0532	3.1876	1.0594	3.0281	43
18	1.0419	3.5629	1.0474	3.3627	1.0533	3.1848	1.0595	3.0256	42
19	1.0420	3.5594	1.0475	3.3596	1.0534	3.1820	1.0596	3.0231	41
20	1.0420	3.5559	1.0476	3.3565	1.0535	3.1792	1.0598	3.0206	40
21	1.0421	3.5523	1.0477	3.3534	1.0536	3.1764	1.0599	3.0181	39
22	1.0422	3.5488	1.0478	3.3502	1.0537	3.1736	1.0600	3.0156	38
23	1.0423	3.5453	1.0478	3.3471	1.0538	3.1708	1.0601	3.0131	37
24	1.0424	3.5418	1.0479	3.3440	1.0539	3.1681	1.0602	3.0106	36
25	1.0425	3.5383	1.0480	3.3409	1.0540	3.1653	1.0603	3.0081	35
26	1.0426	3.5348	1.0481	3.3378	1.0541	3.1625	1.0604	3.0056	34
27	1.0427	3.5313	1.0482	3.3347	1.0542	3.1598	1.0605	3.0031	33
28	1.0428	3.5279	1.0483	3.3316	1.0543	3.1570	1.0606	3.0007	32
29	1.0428	3.5244	1.0484	3.3286	1.0544	3.1543	1.0607	2.9982	31
30	1.0429	3.5209	1.0485	3.3255	1.0545	3.1515	1.0608	2.9957	30
31	1.0430	3.5175	1.0486	3.3224	1.0546	3.1488	1.0609	2.9933	29
32	1.0431	3.5140	1.0487	3.3194	1.0547	3.1461	1.0611	2.9908	28
33	1.0432	3.5106	1.0488	3.3163	1.0548	3.1433	1.0612	2.9884	27
34	1.0433	3.5072	1.0489	3.3133	1.0549	3.1406	1.0613	2.9859	26
35	1.0434	3.5037	1.0490	3.3102	1.0550	3.1379	1.0614	2.9835	25
36	1.0435	3.5003	1.0491	3.3072	1.0551	3.1352	1.0615	2.9810	24
37	1.0436	3.4969	1.0492	3.3042	1.0552	3.1325	1.0616	2.9786	23
38	1.0437	3.4935	1.0493	3.3011	1.0553	3.1298	1.0617	2.9762	22
39	1.0438	3.4901	1.0494	3.2981	1.0554	3.1271	1.0618	2.9738	21
40	1.0438	3.4867	1.0495	3.2951	1.0555	3.1244	1.0619	2.9713	20
41	1.0439	3.4833	1.0496	3.2921	1.0556	3.1217	1.0620	2.9689	19
42	1.0440	3.4799	1.0497	3.2891	1.0557	3.1190	1.0622	2.9665	18
43	1.0441	3.4766	1.0498	3.2861	1.0558	3.1163	1.0623	2.9641	17
44	1.0442	3.4732	1.0499	3.2831	1.0559	3.1137	1.0624	2.9617	16
45	1.0443	3.4698	1.0500	3.2801	1.0560	3.1110	1.0625	2.9593	15
46	1.0444	3.4665	1.0501	3.2772	1.0561	3.1083	1.0626	2.9569	14
47	1.0445	3.4632	1.0502	3.2742	1.0562	3.1057	1.0627	2.9545	13
48	1.0446	3.4598	1.0503	3.2712	1.0563	3.1030	1.0628	2.9521	12
49	1.0447	3.4565	1.0504	3.2683	1.0565	3.1004	1.0629	2.9497	11
50	1.0448	3.4532	1.0505	3.2653	1.0566	3.0977	1.0630	2.9474	10
51	1.0448	3.4498	1.0506	3.2624	1.0567	3.0951	1.0632	2.9450	9
52	1.0449	3.4465	1.0507	3.2594	1.0568	3.0925	1.0633	2.9426	8
53	1.0450	3.4432	1.0508	3.2565	1.0569	3.0898	1.0634	2.9402	7
54	1.0451	3.4399	1.0509	3.2535	1.0570	3.0872	1.0635	2.9379	6
55	1.0452	3.4366	1.0510	3.2506	1.0571	3.0846	1.0636	2.9355	5
56	1.0453	3.4334	1.0511	3.2477	1.0572	3.0820	1.0637	2.9332	4
57	1.0454	3.4301	1.0512	3.2448	1.0573	3.0793	1.0638	2.9308	3
58	1.0455	3.4268	1.0513	3.2419	1.0574	3.0767	1.0639	2.9285	2
59	1.0456	3.4236	1.0514	3.2390	1.0575	3.0741	1.0641	2.9261	1
60	1.0457	3.4203	1.0515	3.2361	1.0576	3.0715	1.0642	2.9238	0
′	cosec 73°	sec 73°	cosec 72°	sec 72°	cosec 71°	sec 71°	cosec 70°	sec 70°	′

′	20°		21°		22°		23°		′
	sec	cosec	sec	cosec	sec	cosec	sec	cosec	
0	1.0642	2.9238	1.0711	2.7904	1.0785	2.6695	1.0864	2.5593	60
1	1.0643	2.9215	1.0713	2.7883	1.0787	2.6675	1.0865	2.5575	59
2	1.0644	2.9191	1.0714	2.7862	1.0788	2.6656	1.0866	2.5558	58
3	1.0645	2.9168	1.0715	2.7841	1.0789	2.6637	1.0868	2.5540	57
4	1.0646	2.9145	1.0716	2.7820	1.0790	2.6618	1.0869	2.5523	56
5	1.0647	2.9122	1.0717	2.7799	1.0792	2.6599	1.0870	2.5506	55
6	1.0648	2.9098	1.0719	2.7778	1.0793	2.6580	1.0872	2.5488	54
7	1.0650	2.9075	1.0720	2.7757	1.0794	2.6561	1.0873	2.5471	53
8	1.0651	2.9052	1.0721	2.7736	1.0795	2.6542	1.0874	2.5453	52
9	1.0652	2.9029	1.0722	2.7715	1.0797	2.6523	1.0876	2.5436	51
10	1.0653	2.9006	1.0723	2.7694	1.0798	2.6504	1.0877	2.5419	50
11	1.0654	2.8983	1.0725	2.7674	1.0799	2.6485	1.0878	2.5402	49
12	1.0655	2.8960	1.0726	2.7653	1.0801	2.6466	1.0880	2.5384	48
13	1.0656	2.8937	1.0727	2.7632	1.0802	2.6447	1.0881	2.5367	47
14	1.0658	2.8915	1.0728	2.7611	1.0803	2.6428	1.0882	2.5350	46
15	1.0659	2.8892	1.0729	2.7591	1.0804	2.6410	1.0884	2.5333	45
16	1.0660	2.8869	1.0731	2.7570	1.0806	2.6391	1.0885	2.5316	44
17	1.0661	2.8846	1.0732	2.7550	1.0807	2.6372	1.0886	2.5299	43
18	1.0662	2.8824	1.0733	2.7529	1.0808	2.6353	1.0888	2.5281	42
19	1.0663	2.8801	1.0734	2.7509	1.0810	2.6335	1.0889	2.5264	41
20	1.0664	2.8778	1.0736	2.7488	1.0811	2.6316	1.0891	2.5247	40
21	1.0666	2.8756	1.0737	2.7468	1.0812	2.6297	1.0892	2.5230	39
22	1.0667	2.8733	1.0738	2.7447	1.0813	2.6279	1.0893	2.5213	38
23	1.0668	2.8711	1.0739	2.7427	1.0815	2.6260	1.0895	2.5196	37
24	1.0669	2.8688	1.0740	2.7406	1.0816	2.6242	1.0896	2.5179	36
25	1.0670	2.8666	1.0742	2.7386	1.0817	2.6223	1.0897	2.5163	35
26	1.0671	2.8644	1.0743	2.7366	1.0819	2.6205	1.0899	2.5146	34
27	1.0673	2.8621	1.0744	2.7346	1.0820	2.6186	1.0900	2.5129	33
28	1.0674	2.8599	1.0745	2.7325	1.0821	2.6168	1.0902	2.5112	32
29	1.0675	2.8577	1.0747	2.7305	1.0823	2.6150	1.0903	2.5095	31
30	1.0676	2.8554	1.0748	2.7285	1.0824	2.6131	1.0904	2.5078	30
31	1.0677	2.8532	1.0749	2.7265	1.0825	2.6113	1.0906	2.5062	29
32	1.0678	2.8510	1.0750	2.7245	1.0826	2.6095	1.0907	2.5045	28
33	1.0679	2.8488	1.0751	2.7225	1.0828	2.6076	1.0908	2.5028	27
34	1.0681	2.8466	1.0753	2.7205	1.0829	2.6058	1.0910	2.5011	26
35	1.0682	2.8444	1.0754	2.7185	1.0830	2.6040	1.0911	2.4995	25
36	1.0683	2.8422	1.0755	2.7165	1.0832	2.6022	1.0913	2.4978	24
37	1.0684	2.8400	1.0756	2.7145	1.0833	2.6003	1.0914	2.4961	23
38	1.0685	2.8378	1.0758	2.7125	1.0834	2.5985	1.0915	2.4945	22
39	1.0686	2.8356	1.0759	2.7105	1.0836	2.5967	1.0917	2.4928	21
40	1.0688	2.8334	1.0760	2.7085	1.0837	2.5949	1.0918	2.4912	20
41	1.0689	2.8312	1.0761	2.7065	1.0838	2.5931	1.0920	2.4895	19
42	1.0690	2.8290	1.0763	2.7045	1.0840	2.5913	1.0921	2.4879	18
43	1.0691	2.8269	1.0764	2.7026	1.0841	2.5895	1.0922	2.4862	17
44	1.0692	2.8247	1.0765	2.7006	1.0842	2.5877	1.0924	2.4846	16
45	1.0694	2.8225	1.0766	2.6986	1.0844	2.5859	1.0925	2.4829	15
46	1.0695	2.8204	1.0768	2.6967	1.0845	2.5841	1.0927	2.4813	14
47	1.0696	2.8182	1.0769	2.6947	1.0846	2.5823	1.0928	2.4797	13
48	1.0697	2.8160	1.0770	2.6927	1.0847	2.5805	1.0929	2.4780	12
49	1.0698	2.8139	1.0771	2.6908	1.0849	2.5787	1.0931	2.4764	11
50	1.0699	2.8117	1.0773	2.6888	1.0850	2.5770	1.0932	2.4748	10
51	1.0701	2.8096	1.0774	2.6869	1.0851	2.5752	1.0934	2.4731	9
52	1.0702	2.8074	1.0775	2.6849	1.0853	2.5734	1.0935	2.4715	8
53	1.0703	2.8053	1.0776	2.6830	1.0854	2.5716	1.0936	2.4699	7
54	1.0704	2.8032	1.0778	2.6810	1.0855	2.5699	1.0938	2.4683	6
55	1.0705	2.8010	1.0779	2.6791	1.0857	2.5681	1.0939	2.4666	5
56	1.0707	2.7989	1.0780	2.6772	1.0858	2.5663	1.0941	2.4650	4
57	1.0708	2.7968	1.0781	2.6752	1.0859	2.5646	1.0942	2.4634	3
58	1.0709	2.7947	1.0783	2.6733	1.0861	2.5628	1.0943	2.4618	2
59	1.0710	2.7925	1.0784	2.6714	1.0862	2.5610	1.0945	2.4602	1
60	1.0711	2.7904	1.0785	2.6695	1.0864	2.5593	1.0946	2.4586	0
′	cosec	sec	cosec	sec	cosec	sec	cosec	sec	′
	69°		68°		67°		66°		

		2.4586	1034	2.3662		2.2812	1223	2.2027	60
	0	2.4570	1035	2.3647	27	2.2798	1225	2.2014	59
	0	2.4554	1037	2.3632	29	2.2784	1226	2.2002	58
	0	2.4538	1038	2.3618	31	2.2771	1228	2.1989	57
	0	2.4522	1040	2.3603	32	2.2757	1230	2.1977	56
	0	2.4506	1041	2.3588	34	2.2744	1231	2.1964	55
	0	2.4490	1043	2.3574	35	2.2730	1233	2.1952	54
	0	2.4474	1044	2.3559	37	2.2717	1235	2.1939	53
	0	2.4458	1046	2.3544	39	2.2703	1237	2.1927	52
	0	2.4442	1047	2.3530	40	2.2690	1238	2.1914	51
		2.4426	049	2.3515	42	2.2676	1240	2.1902	50
	0962	2.4411	050	2.3501	43	2.2663	1242	2.1889	49
	0963	2.4395	052	2.3486	45	2.2650	1243	2.1877	48
1	0965	2.4379	053	2.3472	47	2.2636	1245	2.1865	47
1	0966	2.4363	055	2.3457	48	2.2623	1247	2.1852	46
1	0968	2.4347	056	2.3443	50	2.2610	1248	2.1840	45
1	0969	2.4332	058	2.3428	51	2.2596	1250	2.1828	44
1	0971	2.4316	059	2.3414	53	2.2583	1252	2.1815	43
1	0972	2.4300	061	2.3399	55	2.2570	1253	2.1803	42
1	0973	2.4285	062	2.3385	56	2.2556	1255	2.1791	41
2	0975	2.4269	064	2.3371	58	2.2543	1257	2.1778	40
2	0976	2.42	065	2.3356	59	2.2530	1258	2.1766	39
2	0978	2.423	067	2.3342	61	2.2517	1260	2.1754	38
2	0979	2.422	068	2.3328	63	2.2503	1262	2.1742	37
2	0981	2.420	070	2.3313	64	2.2490	1264	2.1730	36
2		2.419	072	2.3299	66	2.2477	1265	2.1717	35
2		2.417	073	2.3285	67	2.2464	1267	2.1705	34
		2.416	075	2.3271	69	2.2451	1269	2.1693	33
		2.414	076	2.3256	71	2.2438	1270	2.1681	32
	09	2.413	078	2.3242	72	2.2425	1272	2.1669	31
	09 9	2.411	079	2.3228	74	2.2411	1274	2.1657	30
	0991	2.4099	081	2.3214	76	2.2398	1275	2.1645	29
	0992	2.4083	082	2.3200	77	2.2385	1277	2.1633	28
	0994	2.4068	084	2.3186	79	2.2372	1279	2.1620	27
	0995	2.4053	085	2.3172	80	2.2359	1281	2.1608	26
	0997	2.4037	087	2.3158	82	2.2346	1282	2.1596	25
	0998	2.	088	2.3143	84	2.2333	1284	2.1584	24
	1000	2.	090	2.3129	85	2.2320	1286	2.1572	23
	1001	2.	092	2.3115	87	2.2307	1287	2.1560	22
	1003	2.	093	2.3101	89	2.2294	1289	2.1548	21
	1004	2.3961	095	2.3087	90	2.2282	1291	2.1536	20
	005	2.3946	096	2.3073	92	2.2269	1293	2.1525	19
	007	2.3931	098	2.3059	93	2.2256	1294	2.1513	18
	008	2.3916	099	2.3046	95	2.2243	1296	2.1501	17
	010	2.3901	101	2.3032	97	2.2230	1298	2.1489	16
	011	2.3886	102	2.3018	98	2.2217	1299	2.1477	15
	013	2.3871	104	2.3004	00	2.2204	1301	2.1465	14
	014	2.3856	106	2.2990	02	2.2192	1303	2.1453	13
	016	2.3841	107	2.2976	03	2.2179	1305	2.1441	12
	017	2.3826	109	2.2962	5	2.2166	1306	2.1430	11
	019	2.3811	110	2.2949	7	2.2153	1308	2.1418	10
	020	2.3796	1112	2.2935	08	2.2141	1310	2.1406	9
	022	2.3781	1113	2.2921	10	2.2128	1312	2.1394	8
	023	2.3766	1115	2.2907	12	2.2115	1313	2.1382	7
	025	2.3751	1116	2.2894	13	2.2103	1315	2.1371	6
	026	2.3736	1118	2.2880	15	2.2090	1317	2.1359	5
	028	2.3721	1120	2.2866	7	2.2077	1319	2.1347	4
	029	2.3706	1121	2.2853	8	2.2065	1320	2.1335	3
	031	2.3691	1123	2.2839	0	2.2052	1322	2.1324	2
	032	2.3677	1124	2.2825	2	2.2039	1324	2.1312	1
		2.3662	1126	2.2812	3	2.2027	1326	2.1300	0

65°

′	28°		29°		30°		31°		′
	sec	cosec	sec	cosec	sec	cosec	sec	cosec	
0	1.1326	2.1300	1.1433	2.0627	1.1547	2.0000	1.1666	1.9416	60
1	1.1327	2.1289	1.1435	2.0616	1.1549	1.9990	1.1668	1.9407	59
2	1.1329	2.1277	1.1437	2.0605	1.1551	1.9980	1.1670	1.9397	58
3	1.1331	2.1266	1.1439	2.0594	1.1553	1.9970	1.1672	1.9388	57
4	1.1333	2.1254	1.1441	2.0583	1.1555	1.9960	1.1674	1.9378	56
5	1.1334	2.1242	1.1443	2.0573	1.1557	1.9950	1.1676	1.9369	55
6	1.1336	2.1231	1.1445	2.0562	1.1559	1.9940	1.1678	1.9360	54
7	1.1338	2.1219	1.1446	2.0551	1.1561	1.9930	1.1681	1.9350	53
8	1.1340	2.1208	1.1448	2.0540	1.1562	1.9920	1.1683	1.9341	52
9	1.1341	2.1196	1.1450	2.0530	1.1564	1.9910	1.1685	1.9332	51
10	1.1343	2.1185	1.1452	2.0519	1.1566	1.9900	1.1687	1.9322	50
11	1.1345	2.1173	1.1454	2.0508	1.1568	1.9890	1.1689	1.9313	49
12	1.1347	2.1162	1.1456	2.0498	1.1570	1.9880	1.1691	1.9304	48
13	1.1349	2.1150	1.1458	2.0487	1.1572	1.9870	1.1693	1.9295	47
14	1.1350	2.1139	1.1459	2.0476	1.1574	1.9860	1.1695	1.9285	46
15	1.1352	2.1127	1.1461	2.0466	1.1576	1.9850	1.1697	1.9276	45
16	1.1354	2.1116	1.1463	2.0455	1.1578	1.9840	1.1699	1.9267	44
17	1.1356	2.1104	1.1465	2.0444	1.1580	1.9830	1.1701	1.9258	43
18	1.1357	2.1093	1.1467	2.0434	1.1582	1.9820	1.1703	1.9248	42
19	1.1359	2.1082	1.1469	2.0423	1.1584	1.9811	1.1705	1.9239	41
20	1.1361	2.1070	1.1471	2.0413	1.1586	1.9801	1.1707	1.9230	40
21	1.1363	2.1059	1.1473	2.0402	1.1588	1.9791	1.1709	1.9221	39
22	1.1365	2.1048	1.1474	2.0392	1.1590	1.9781	1.1712	1.9212	38
23	1.1366	2.1036	1.1476	2.0381	1.1592	1.9771	1.1714	1.9203	37
24	1.1368	2.1025	1.1478	2.0370	1.1594	1.9761	1.1716	1.9193	36
25	1.1370	2.1014	1.1480	2.0360	1.1596	1.9752	1.1718	1.9184	35
26	1.1372	2.1002	1.1482	2.0349	1.1598	1.9742	1.1720	1.9175	34
27	1.1373	2.0991	1.1484	2.0339	1.1600	1.9732	1.1722	1.9166	33
28	1.1375	2.0980	1.1486	2.0329	1.1602	1.9722	1.1724	1.9157	32
29	1.1377	2.0969	1.1488	2.0318	1.1604	1.9713	1.1726	1.9148	31
30	1.1379	2.0957	1.1489	2.0308	1.1606	1.9703	1.1728	1.9139	30
31	1.1381	2.0946	1.1491	2.0297	1.1608	1.9693	1.1730	1.9130	29
32	1.1382	2.0935	1.1493	2.0287	1.1610	1.9683	1.1732	1.9121	28
33	1.1384	2.0924	1.1495	2.0276	1.1612	1.9674	1.1734	1.9112	27
34	1.1386	2.0912	1.1497	2.0266	1.1614	1.9664	1.1737	1.9102	26
35	1.1388	2.0901	1.1499	2.0256	1.1616	1.9654	1.1739	1.9093	25
36	1.1390	2.0890	1.1501	2.0245	1.1618	1.9645	1.1741	1.9084	24
37	1.1391	2.0879	1.1503	2.0235	1.1620	1.9635	1.1743	1.9075	23
38	1.1393	2.0868	1.1505	2.0224	1.1622	1.9625	1.1745	1.9066	22
39	1.1395	2.0857	1.1507	2.0214	1.1624	1.9616	1.1747	1.9057	21
40	1.1397	2.0846	1.1508	2.0204	1.1626	1.9606	1.1749	1.9048	20
41	1.1399	2.0835	1.1510	2.0194	1.1628	1.9596	1.1751	1.9039	19
42	1.1401	2.0824	1.1512	2.0183	1.1630	1.9587	1.1753	1.9030	18
43	1.1402	2.0812	1.1514	2.0173	1.1632	1.9577	1.1756	1.9021	17
44	1.1404	2.0801	1.1516	2.0163	1.1634	1.9568	1.1758	1.9013	16
45	1.1406	2.0790	1.1518	2.0152	1.1636	1.9558	1.1760	1.9004	15
46	1.1408	2.0779	1.1520	2.0142	1.1638	1.9549	1.1762	1.8995	14
47	1.1410	2.0768	1.1522	2.0132	1.1640	1.9539	1.1764	1.8986	13
48	1.1411	2.0757	1.1524	2.0122	1.1642	1.9530	1.1766	1.8977	12
49	1.1413	2.0746	1.1526	2.0111	1.1644	1.9520	1.1768	1.8968	11
50	1.1415	2.0735	1.1528	2.0101	1.1646	1.9510	1.1770	1.8959	10
51	1.1417	2.0725	1.1530	2.0091	1.1648	1.9501	1.1772	1.8950	9
52	1.1419	2.0714	1.1531	2.0081	1.1650	1.9491	1.1775	1.8941	8
53	1.1421	2.0703	1.1533	2.0071	1.1652	1.9482	1.1777	1.8932	7
54	1.1422	2.0692	1.1535	2.0061	1.1654	1.9473	1.1779	1.8924	6
55	1.1424	2.0681	1.1537	2.0050	1.1656	1.9463	1.1781	1.8915	5
56	1.1426	2.0670	1.1539	2.0040	1.1658	1.9454	1.1783	1.8906	4
57	1.1428	2.0659	1.1541	2.0030	1.1660	1.9444	1.1785	1.8897	3
58	1.1430	2.0648	1.1543	2.0020	1.1662	1.9435	1.1787	1.8888	2
59	1.1432	2.0637	1.1545	2.0010	1.1664	1.9425	1.1790	1.8879	1
60	1.1433	2.0627	1.1547	2.0000	1.1666	1.9416	1.1792	1.8871	0
′	cosec	sec	cosec	sec	cosec	sec	cosec	sec	′
	61°		60°		59°		58°		

′	32°		33°		34°		35°		′
	sec	cosec	sec	cosec	sec	cosec	sec	cosec	
0	1.1792	1.8871	1.1924	1.8361	1.2062	1.7883	1.2208	1.7434	60
1	1.1794	1.8862	1.1926	1.8352	1.2064	1.7875	1.2210	1.7427	59
2	1.1796	1.8853	1.1928	1.8344	1.2067	1.7867	1.2213	1.7420	58
3	1.1798	1.8844	1.1930	1.8336	1.2069	1.7860	1.2215	1.7413	57
4	1.1800	1.8836	1.1933	1.8328	1.2072	1.7852	1.2218	1.7405	56
5	1.1802	1.8827	1.1935	1.8320	1.2074	1.7844	1.2220	1.7398	55
6	1.1805	1.8818	1.1937	1.8311	1.2076	1.7837	1.2223	1.7391	54
7	1.1807	1.8809	1.1939	1.8303	1.2079	1.7829	1.2225	1.7384	53
8	1.1809	1.8801	1.1942	1.8295	1.2081	1.7821	1.2228	1.7377	52
9	1.1811	1.8792	1.1944	1.8287	1.2083	1.7814	1.2230	1.7369	51
10	1.1813	1.8783	1.1946	1.8279	1.2086	1.7806	1.2233	1.7362	50
11	1.1815	1.8775	1.1948	1.8271	1.2088	1.7798	1.2235	1.7355	49
12	1.1818	1.8766	1.1951	1.8263	1.2091	1.7791	1.2238	1.7348	48
13	1.1820	1.8757	1.1953	1.8255	1.2093	1.7783	1.2240	1.7341	47
14	1.1822	1.8749	1.1955	1.8246	1.2095	1.7776	1.2243	1.7334	46
15	1.1824	1.8740	1.1958	1.8238	1.2098	1.7768	1.2245	1.7327	45
16	1.1826	1.8731	1.1960	1.8230	1.2100	1.7760	1.2248	1.7319	44
17	1.1828	1.8723	1.1962	1.8222	1.2103	1.7753	1.2250	1.7312	43
18	1.1831	1.8714	1.1964	1.8214	1.2105	1.7745	1.2253	1.7305	42
19	1.1833	1.8706	1.1967	1.8206	1.2107	1.7738	1.2255	1.7298	41
20	1.1835	1.8697	1.1969	1.8198	1.2110	1.7730	1.2258	1.7291	40
21	1.1837	1.8688	1.1971	1.8190	1.2112	1.7723	1.2260	1.7284	39
22	1.1839	1.8680	1.1974	1.8182	1.2115	1.7715	1.2263	1.7277	38
23	1.1841	1.8671	1.1976	1.8174	1.2117	1.7708	1.2265	1.7270	37
24	1.1844	1.8663	1.1978	1.8166	1.2119	1.7700	1.2268	1.7263	36
25	1.1846	1.8654	1.1980	1.8158	1.2122	1.7693	1.2270	1.7256	35
26	1.1848	1.8646	1.1983	1.8150	1.2124	1.7685	1.2273	1.7249	34
27	1.1850	1.8637	1.1985	1.8142	1.2127	1.7678	1.2276	1.7242	33
28	1.1852	1.8629	1.1987	1.8134	1.2129	1.7670	1.2278	1.7234	32
29	1.1855	1.8620	1.1990	1.8126	1.2132	1.7663	1.2281	1.7227	31
30	1.1857	1.8611	1.1992	1.8118	1.2134	1.7655	1.2283	1.7220	30
31	1.1859	1.8603	1.1994	1.8110	1.2136	1.7648	1.2286	1.7213	29
32	1.1861	1.8595	1.1997	1.8102	1.2139	1.7640	1.2288	1.7206	28
33	1.1863	1.8586	1.1999	1.8094	1.2141	1.7633	1.2291	1.7199	27
34	1.1866	1.8578	1.2001	1.8086	1.2144	1.7625	1.2293	1.7192	26
35	1.1868	1.8569	1.2004	1.8078	1.2146	1.7618	1.2296	1.7185	25
36	1.1870	1.8561	1.2006	1.8070	1.2149	1.7610	1.2298	1.7178	24
37	1.1872	1.8552	1.2008	1.8062	1.2151	1.7603	1.2301	1.7171	23
38	1.1874	1.8544	1.2010	1.8054	1.2153	1.7596	1.2304	1.7164	22
39	1.1877	1.8535	1.2013	1.8047	1.2156	1.7588	1.2306	1.7157	21
40	1.1879	1.8527	1.2015	1.8039	1.2158	1.7581	1.2309	1.7151	20
41	1.1881	1.8519	1.2017	1.8031	1.2161	1.7573	1.2311	1.7144	19
42	1.1883	1.8510	1.2020	1.8023	1.2163	1.7566	1.2314	1.7137	18
43	1.1886	1.8502	1.2022	1.8015	1.2166	1.7559	1.2316	1.7130	17
44	1.1888	1.8493	1.2024	1.8007	1.2168	1.7551	1.2319	1.7123	16
45	1.1890	1.8485	1.2027	1.7999	1.2171	1.7544	1.2322	1.7116	15
46	1.1892	1.8477	1.2029	1.7992	1.2173	1.7537	1.2324	1.7109	14
47	1.1894	1.8468	1.2031	1.7984	1.2175	1.7529	1.2327	1.7102	13
48	1.1897	1.8460	1.2034	1.7976	1.2178	1.7522	1.2329	1.7095	12
49	1.1899	1.8452	1.2036	1.7968	1.2180	1.7514	1.2332	1.7088	11
50	1.1901	1.8443	1.2039	1.7960	1.2183	1.7507	1.2335	1.7081	10
51	1.1903	1.8435	1.2041	1.7953	1.2185	1.7500	1.2337	1.7075	9
52	1.1906	1.8427	1.2043	1.7945	1.2188	1.7493	1.2340	1.7068	8
53	1.1908	1.8418	1.2046	1.7937	1.2190	1.7485	1.2342	1.7061	7
54	1.1910	1.8410	1.2048	1.7929	1.2193	1.7478	1.2345	1.7054	6
55	1.1912	1.8402	1.2050	1.7921	1.2195	1.7471	1.2348	1.7047	5
56	1.1915	1.8394	1.2053	1.7914	1.2198	1.7463	1.2350	1.7040	4
57	1.1917	1.8385	1.2055	1.7906	1.2200	1.7456	1.2353	1.7033	3
58	1.1919	1.8377	1.2057	1.7898	1.2203	1.7449	1.2355	1.7027	2
59	1.1921	1.8369	1.2060	1.7891	1.2205	1.7442	1.2358	1.7020	1
60	1.1924	1.8361	1.2062	1.7883	1.2208	1.7434	1.2361	1.7013	0
′	cosec	sec	cosec	sec	cosec	sec	cosec	sec	′
	57°		56°		55°		54°		

′	36° sec	36° cosec	37° sec	37° cosec	38° sec	38° cosec	39° sec	39° cosec	′
0	1.2361	1.7013	1.2521	1.6616	1.2690	1.6243	1.2867	1.5890	60
1	1.2363	1.7006	1.2524	1.6610	1.2693	1.6237	1.2871	1.5884	59
2	1.2366	1.6999	1.2527	1.6603	1.2696	1.6231	1.2874	1.5879	58
3	1.2368	1.6993	1.2530	1.6597	1.2699	1.6224	1.2877	1.5873	57
4	1.2371	1.6986	1.2532	1.6591	1.2702	1.6218	1.2880	1.5867	56
5	1.2374	1.6979	1.2535	1.6584	1.2705	1.6212	1.2883	1.5862	55
6	1.2376	1.6972	1.2538	1.6578	1.2707	1.6206	1.2886	1.5856	54
7	1.2379	1.6965	1.2541	1.6572	1.2710	1.6200	1.2889	1.5850	53
8	1.2382	1.6959	1.2543	1.6565	1.2713	1.6194	1.2892	1.5845	52
9	1.2384	1.6952	1.2546	1.6559	1.2716	1.6188	1.2895	1.5839	51
10	1.2387	1.6945	1.2549	1.6552	1.2719	1.6182	1.2898	1.5833	50
11	1.2389	1.6938	1.2552	1.6546	1.2722	1.6176	1.2901	1.5828	49
12	1.2392	1.6932	1.2554	1.6540	1.2725	1.6170	1.2904	1.5822	48
13	1.2395	1.6925	1.2557	1.6533	1.2728	1.6164	1.2907	1.5816	47
14	1.2397	1.6918	1.2560	1.6527	1.2731	1.6159	1.2910	1.5811	46
15	1.2400	1.6912	1.2563	1.6521	1.2734	1.6153	1.2913	1.5805	45
16	1.2403	1.6905	1.2565	1.6514	1.2737	1.6147	1.2916	1.5799	44
17	1.2405	1.6898	1.2568	1.6508	1.2739	1.6141	1.2919	1.5794	43
18	1.2408	1.6891	1.2571	1.6502	1.2742	1.6135	1.2922	1.5788	42
19	1.2411	1.6885	1.2574	1.6496	1.2745	1.6129	1.2926	1.5783	41
20	1.2413	1.6878	1.2577	1.6489	1.2748	1.6123	1.2929	1.5777	40
21	1.2416	1.6871	1.2579	1.6483	1.2751	1.6117	1.2932	1.5771	39
22	1.2419	1.6865	1.2582	1.6477	1.2754	1.6111	1.2935	1.5766	38
23	1.2421	1.6858	1.2585	1.6470	1.2757	1.6105	1.2938	1.5760	37
24	1.2424	1.6851	1.2588	1.6464	1.2760	1.6099	1.2941	1.5755	36
25	1.2427	1.6845	1.2591	1.6458	1.2763	1.6093	1.2944	1.5749	35
26	1.2429	1.6838	1.2593	1.6452	1.2766	1.6087	1.2947	1.5743	34
27	1.2432	1.6831	1.2596	1.6445	1.2769	1.6081	1.2950	1.5738	33
28	1.2435	1.6825	1.2599	1.6439	1.2772	1.6077	1.2953	1.5732	32
29	1.2437	1.6818	1.2602	1.6433	1.2775	1.6070	1.2956	1.5727	31
30	1.2440	1.6812	1.2605	1.6427	1.2778	1.6064	1.2960	1.5721	30
31	1.2443	1.6805	1.2607	1.6420	1.2781	1.6058	1.2963	1.5716	29
32	1.2445	1.6798	1.2610	1.6414	1.2784	1.6052	1.2966	1.5710	28
33	1.2448	1.6792	1.2613	1.6408	1.2787	1.6046	1.2969	1.5705	27
34	1.2451	1.6785	1.2616	1.6402	1.2790	1.6040	1.2972	1.5699	26
35	1.2453	1.6779	1.2619	1.6396	1.2793	1.6034	1.2975	1.5694	25
36	1.2456	1.6772	1.2622	1.6389	1.2795	1.6029	1.2978	1.5688	24
37	1.2459	1.6766	1.2624	1.6383	1.2798	1.6023	1.2981	1.5683	23
38	1.2461	1.6759	1.2627	1.6377	1.2801	1.6017	1.2985	1.5677	22
39	1.2464	1.6752	1.2630	1.6371	1.2804	1.6011	1.2988	1.5672	21
40	1.2467	1.6746	1.2633	1.6365	1.2807	1.6005	1.2991	1.5666	20
41	1.2470	1.6739	1.2636	1.6359	1.2810	1.6000	1.2994	1.5661	19
42	1.2472	1.6733	1.2639	1.6352	1.2813	1.5994	1.2997	1.5655	18
43	1.2475	1.6726	1.2641	1.6346	1.2816	1.5988	1.3000	1.5650	17
44	1.2478	1.6720	1.2644	1.6340	1.2819	1.5982	1.3003	1.5644	16
45	1.2480	1.6713	1.2647	1.6334	1.2822	1.5976	1.3006	1.5639	15
46	1.2483	1.6707	1.2650	1.6328	1.2825	1.5971	1.3010	1.5633	14
47	1.2486	1.6700	1.2653	1.6322	1.2828	1.5965	1.3013	1.5628	13
48	1.2488	1.6694	1.2656	1.6316	1.2831	1.5959	1.3016	1.5622	12
49	1.2490	1.6687	1.2659	1.6309	1.2834	1.5953	1.3019	1.5617	11
50	1.2494	1.6681	1.2661	1.6303	1.2837	1.5947	1.3022	1.5611	10
51	1.2497	1.6674	1.2664	1.6297	1.2840	1.5942	1.3025	1.5606	9
52	1.2499	1.6668	1.2667	1.6291	1.2843	1.5936	1.3029	1.5600	8
53	1.2502	1.6661	1.2670	1.6285	1.2846	1.5930	1.3032	1.5595	7
54	1.2505	1.6655	1.2673	1.6279	1.2849	1.5924	1.3035	1.5590	6
55	1.2508	1.6648	1.2676	1.6273	1.2852	1.5919	1.3038	1.5584	5
56	1.2510	1.6642	1.2679	1.6267	1.2855	1.5913	1.3041	1.5579	4
57	1.2513	1.6636	1.2681	1.6261	1.2858	1.5907	1.3044	1.5573	3
58	1.2516	1.6629	1.2684	1.6255	1.2861	1.5901	1.3048	1.5568	2
59	1.2519	1.6623	1.2687	1.6249	1.2864	1.5896	1.3051	1.5563	1
60	1.2521	1.6616	1.2690	1.6243	1.2867	1.5890	1.3054	1.5557	0
′	53° cosec	53° sec	52° cosec	52° sec	51° cosec	51° sec	50° cosec	50° sec	′

′	40° sec	40° cosec	41° sec	41° cosec	42° sec	42° cosec	43° sec	43° cosec	′
0	1.3054	1.5557	1.3250	1.5242	1.3456	1.4945	1.3673	1.4663	60
1	1.3057	1.5552	1.3253	1.5237	1.3460	1.4940	1.3677	1.4658	59
2	1.3060	1.5546	1.3257	1.5232	1.3463	1.4935	1.3681	1.4654	58
3	1.3064	1.5541	1.3260	1.5227	1.3467	1.4930	1.3684	1.4649	57
4	1.3067	1.5536	1.3263	1.5222	1.3470	1.4925	1.3688	1.4644	56
5	1.3070	1.5530	1.3267	1.5217	1.3474	1.4921	1.3692	1.4640	55
6	1.3073	1.5525	1.3270	1.5212	1.3477	1.4916	1.3695	1.4635	54
7	1.3076	1.5520	1.3274	1.5207	1.3481	1.4911	1.3699	1.4631	53
8	1.3080	1.5514	1.3277	1.5202	1.3485	1.4906	1.3703	1.4626	52
9	1.3083	1.5509	1.3280	1.5197	1.3488	1.4901	1.3707	1.4622	51
10	1.3086	1.5503	1.3284	1.5192	1.3492	1.4897	1.3710	1.4617	50
11	1.3089	1.5498	1.3287	1.5187	1.3495	1.4892	1.3714	1.4613	49
12	1.3092	1.5493	1.3290	1.5182	1.3499	1.4887	1.3718	1.4608	48
13	1.3096	1.5487	1.3294	1.5177	1.3502	1.4882	1.3722	1.4604	47
14	1.3099	1.5482	1.3297	1.5171	1.3506	1.4877	1.3725	1.4599	46
15	1.3102	1.5477	1.3301	1.5166	1.3509	1.4873	1.3729	1.4595	45
16	1.3105	1.5471	1.3304	1.5161	1.3513	1.4868	1.3733	1.4590	44
17	1.3109	1.5466	1.3307	1.5156	1.3517	1.4863	1.3737	1.4586	43
18	1.3112	1.5461	1.3311	1.5151	1.3520	1.4858	1.3740	1.4581	42
19	1.3115	1.5456	1.3314	1.5146	1.3524	1.4854	1.3744	1.4577	41
20	1.3118	1.5450	1.3318	1.5141	1.3527	1.4849	1.3748	1.4572	40
21	1.3121	1.5445	1.3321	1.5136	1.3531	1.4844	1.3752	1.4568	39
22	1.3125	1.5440	1.3324	1.5131	1.3534	1.4839	1.3756	1.4563	38
23	1.3128	1.5434	1.3328	1.5126	1.3538	1.4835	1.3759	1.4559	37
24	1.3131	1.5429	1.3331	1.5121	1.3542	1.4830	1.3763	1.4554	36
25	1.3134	1.5424	1.3335	1.5116	1.3545	1.4825	1.3767	1.4550	35
26	1.3138	1.5419	1.3338	1.5111	1.3549	1.4821	1.3771	1.4545	34
27	1.3141	1.5413	1.3342	1.5106	1.3552	1.4816	1.3774	1.4541	33
28	1.3144	1.5408	1.3345	1.5101	1.3556	1.4811	1.3778	1.4536	32
29	1.3148	1.5403	1.3348	1.5096	1.3560	1.4806	1.3782	1.4532	31
30	1.3151	1.5398	1.3352	1.5092	1.3563	1.4802	1.3786	1.4527	30
31	1.3154	1.5392	1.3355	1.5087	1.3567	1.4797	1.3790	1.4523	29
32	1.3157	1.5387	1.3359	1.5082	1.3571	1.4792	1.3794	1.4518	28
33	1.3161	1.5382	1.3362	1.5077	1.3574	1.4788	1.3797	1.4514	27
34	1.3164	1.5377	1.3366	1.5072	1.3578	1.4783	1.3801	1.4510	26
35	1.3167	1.5371	1.3369	1.5067	1.3581	1.4778	1.3805	1.4505	25
36	1.3170	1.5366	1.3372	1.5062	1.3585	1.4774	1.3809	1.4501	24
37	1.3174	1.5361	1.3376	1.5057	1.3589	1.4769	1.3813	1.4496	23
38	1.3177	1.5356	1.3379	1.5052	1.3592	1.4764	1.3816	1.4492	22
39	1.3180	1.5351	1.3383	1.5047	1.3596	1.4760	1.3820	1.4487	21
40	1.3184	1.5345	1.3386	1.5042	1.3600	1.4755	1.3824	1.4483	20
41	1.3187	1.5340	1.3390	1.5037	1.3603	1.4750	1.3828	1.4479	19
42	1.3190	1.5335	1.3393	1.5032	1.3607	1.4746	1.3832	1.4474	18
43	1.3193	1.5330	1.3397	1.5027	1.3611	1.4741	1.3836	1.4470	17
44	1.3197	1.5325	1.3400	1.5022	1.3614	1.4736	1.3839	1.4465	16
45	1.3200	1.5319	1.3404	1.5018	1.3618	1.4732	1.3843	1.4461	15
46	1.3203	1.5314	1.3407	1.5013	1.3622	1.4727	1.3847	1.4457	14
47	1.3207	1.5309	1.3411	1.5008	1.3625	1.4723	1.3851	1.4452	13
48	1.3210	1.5304	1.3414	1.5003	1.3629	1.4718	1.3855	1.4448	12
49	1.3213	1.5299	1.3418	1.4998	1.3633	1.4713	1.3859	1.4443	11
50	1.3217	1.5294	1.3421	1.4993	1.3636	1.4709	1.3863	1.4439	10
51	1.3220	1.5289	1.3425	1.4988	1.3640	1.4704	1.3867	1.4435	9
52	1.3223	1.5283	1.3428	1.4983	1.3644	1.4699	1.3870	1.4430	8
53	1.3227	1.5278	1.3432	1.4979	1.3647	1.4695	1.3874	1.4426	7
54	1.3230	1.5273	1.3435	1.4974	1.3651	1.4690	1.3878	1.4422	6
55	1.3233	1.5268	1.3439	1.4969	1.3655	1.4686	1.3882	1.4417	5
56	1.3237	1.5263	1.3442	1.4964	1.3658	1.4681	1.3886	1.4413	4
57	1.3340	1.5258	1.3446	1.4959	1.3662	1.4676	1.3890	1.4408	3
58	1.3243	1.5253	1.3449	1.4954	1.3666	1.4672	1.3894	1.4404	2
59	1.3247	1.5248	1.3453	1.4949	1.3669	1.4667	1.3898	1.4400	1
60	1.3250	1.5242	1.3456	1.4945	1.3673	1.4663	1.3902	1.4395	0
′	cosec 49°	sec 49°	cosec 48°	sec 48°	cosec 47°	sec 47°	cosec 46°	sec 46°	′

′	44° sec	44° cosec	′	′	44° sec	44° cosec	′	′	44° sec	44° cosec	′
0	1.3902	1.4395	60	21	1.3984	1.4305	39	41	1.4065	1.4221	19
1	1.3905	1.4391	59	22	1.3988	1.4301	38	42	1.4069	1.4217	18
2	1.3909	1.4387	58	23	1.3992	1.4297	37	43	1.4073	1.4212	17
3	1.3913	1.4382	57	24	1.3996	1.4292	36	44	1.4077	1.4208	16
4	1.3917	1.4378	56	25	1.4000	1.4288	35	45	1.4081	1.4204	15
5	1.3921	1.4374	55	26	1.4004	1.4284	34	46	1.4085	1.4200	14
6	1.3925	1.4370	54	27	1.4008	1.4280	33	47	1.4089	1.4196	13
7	1.3929	1.4365	53	28	1.4012	1.4276	32	48	1.4093	1.4192	12
8	1.3933	1.4361	52	29	1.4016	1.4271	31	49	1.4097	1.4188	11
9	1.3937	1.4357	51	30	1.4020	1.4267	30	50	1.4101	1.4183	10
10	1.3941	1.4352	50								
				31	1.4024	1.4263	29	51	1.4105	1.4179	9
11	1.3945	1.4348	49	32	1.4028	1.4259	28	52	1.4109	1.4175	8
12	1.3949	1.4344	48	33	1.4032	1.4254	27	53	1.4113	1.4171	7
13	1.3953	1.4339	47	34	1.4036	1.4250	26	54	1.4117	1.4167	6
14	1.3957	1.4335	46	35	1.4040	1.4246	25	55	1.4122	1.4163	5
15	1.3960	1.4331	45	36	1.4044	1.4242	24	56	1.4126	1.4159	4
16	1.3964	1.4327	44	37	1.4048	1.4238	23	57	1.4130	1.4154	3
17	1.3968	1.4322	43	38	1.4052	1.4233	22	58	1.4134	1.4150	2
18	1.3972	1.4318	42	39	1.4056	1.4229	21	59	1.4138	1.4146	1
19	1.3976	1.4314	41	40	1.4060	1.4225	20	60	1.4142	1.4142	0
20	1.3980	1.4310	40								
′	45° cosec	45° sec	′	′	45° cosec	45° sec	′	′	45° cosec	45° sec	′

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